CHAPTER 2

THEORY

2.1 Sound Scattering from Two Concentric Fluid Spheres

In order to verify the ray tracing code, the solution to scattering of plane waves by two concentric fluid spheres is solved. For computational simplicity, the spheres are located at the origin of a spherical coordinate system (r, θ, ϕ) as shown in Fig. 2.1. The source is either a plane wave propagating in the $-z$ direction or a point source located at (0, 0, R). Both source placements eliminate any dependence on ϕ.

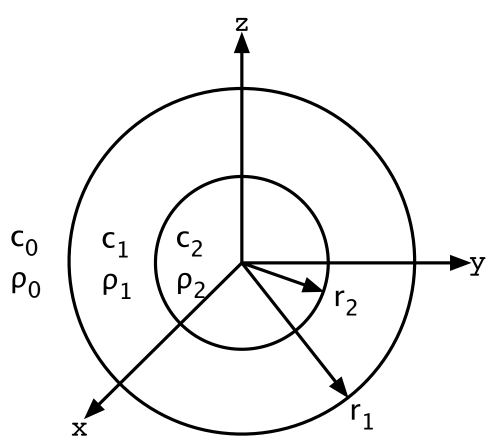

Figure 2.1: Concentric fluid sphere geometry. Infinite medium with density ρ_0, sound speed c_0, and absorption coefficient α_0. The outer sphere has density ρ_1, sound speed c_1, absorption coefficient α_1, and radius r_1. The inner sphere has density ρ_2, sound speed c_2, absorption coefficient α_2, and radius r_2. The spheres are centered at the origin of a spherical coordinate system (r, θ, ϕ) where r is the radial coordinate, θ the azimuthal coordinate, and ϕ the polar coordinate.

The pressure in the medium, p_0, is the sum of the incident pressure p_{0i}, and the scattered pressure p_{0r} [6].

$$p_0 = p_{0i} + p_{0r} \tag{2.1}$$

$$p_{0i} = \mathscr{P}_0 \sum_{m=0}^{\infty} (2m+1)\, \mathscr{L}_m P_m(\mu)\, j_m\left((k_0 + i\alpha_0)r\right) e^{-i\omega t} \tag{2.2}$$

$$p_{0r} = \sum_{m=0}^{\infty} A_m P_m(\mu)\, h_m^{(1)}\left((k_0 + i\alpha_0)r\right) e^{-i\omega t} \tag{2.3}$$

where $\mu = \cos(\theta)$ and $\mathscr{L}_m$ is given by [7]

$$\mathscr{L}_m = \begin{cases} (-i)^m, & \text{plane wave} \\ h_m^{(1)}\left((k_0 + i\alpha_0)R\right), & \text{monopole} \end{cases} \tag{2.4}$$

The pressure in the outer sphere, p_1, is the sum of an outward traveling wave p_{1r} and an inward traveling wave p_{1i}.

$$p_1 = p_{1i} + p_{1r} \tag{2.5}$$

$$p_{1i} = \sum_{m=0}^{\infty} B_m P_m(\mu)\, h_m^{(2)}\left((k_1 + i\alpha_1)r\right) e^{-i\omega t} \tag{2.6}$$

$$p_{1r} = \sum_{m=0}^{\infty} C_m P_m(\mu)\, h_m^{(1)}\left((k_1 + i\alpha_1)r\right) e^{-i\omega t} \tag{2.7}$$

The pressure in the inner sphere can be written as

$$p_{2i} = \sum_{m=0}^{\infty} D_m P_m(\mu)\, j_m\left((k_2 + i\alpha_2)r\right) e^{-i\omega t} \tag{2.8}$$

Four boundary conditions are applicable to the current problem. The first involves the pressure at the boundary between the outer sphere and the medium:

$$p_{0i}(r_1) + p_{0r}(r_1) = p_{1i}(r_1) + p_{1r}(r_1) \tag{2.9}$$

The second boundary condition involves the pressure at the boundary between the inner and outer spheres:

$$p_{1i}(r_2) + p_{1r}(r_2) = p_{2i}(r_2) \tag{2.10}$$

The third boundary condition involves the radial velocity at the boundary between the outer sphere and the medium:

$$u_{0,rad}\left(r_1\right) = u_{1,rad}\left(r_1\right) \tag{2.11}$$

The final boundary condition involves the radial velocity at the boundary between the inner and outer spheres:

$$u_{1,rad}\left(r_2\right) = u_{2,rad}\left(r_2\right) \tag{2.12}$$

For the time harmonic case, the radial velocity becomes

$$u_{rad} = -\left(\frac{i}{Z_n}\right)p' \tag{2.13}$$

where the $'$ symbol denotes derivative with respect to the total argument and

$$Z_n = \frac{\rho_n c_n}{1 + i\frac{\alpha c_n}{\omega}} \tag{2.14}$$

The radial velocities then become

$$u_{0i} = -\frac{i}{Z_0}\mathscr{P}_0 \sum_{m=0}^{\infty} (2m+1)\,\mathscr{L}_m P_m\left(\mu\right) j_m'\left(\tilde{k}_0 r\right) e^{-i\omega t} \tag{2.15}$$

$$u_{0r} = -\frac{i}{Z_0}\sum_{m=0}^{\infty} A_m P_m\left(\mu\right) h_m^{(1)\prime}\left(\tilde{k}_0 r\right) e^{-i\omega t} \tag{2.16}$$

$$u_{1i} = -\frac{i}{Z_1}\sum_{m=0}^{\infty} B_m P_m\left(\mu\right) h_m^{(2)\prime}\left(\tilde{k}_1 r\right) e^{-i\omega t} \tag{2.17}$$

$$u_{1r} = -\frac{i}{Z_1}\sum_{m=0}^{\infty} C_m P_m\left(\mu\right) h_m^{(1)\prime}\left(\tilde{k}_1 r\right) e^{-i\omega t} \tag{2.18}$$

$$u_{2i} = -\frac{i}{Z_2}\sum_{m=0}^{\infty} D_m P_m\left(\mu\right) j_m'\left(\tilde{k}_2 r\right) e^{-i\omega t} \tag{2.19}$$

where $\tilde{k}_n = k_n + i\alpha_n$. The boundary conditions result in the following system of equations:

$$\frac{\mathscr{P}_0}{Z_0}(2m+1)\mathscr{L}_m j'_m\left(\tilde{k}_0 r_1\right) + \frac{1}{Z_0} A_m h^{(1)\prime}_m\left(\tilde{k}_0 r_1\right) = \frac{1}{Z_1} B_m h^{(2)\prime}_m\left(\tilde{k}_1 r_1\right) + \frac{1}{Z_1} C_m h^{(1)\prime}_m\left(\tilde{k}_1 r_1\right) \quad (2.20)$$

$$\frac{1}{Z_1} B_m h^{(2)\prime}_m\left(\tilde{k}_1 r_2\right) + \frac{1}{Z_1} C_m h^{(1)\prime}_m\left(\tilde{k}_1 r_2\right) = \frac{1}{Z_2} D_m j'_m\left(\tilde{k}_2 r_2\right) \quad (2.21)$$

$$\mathscr{P}_0(2m+1)\mathscr{L}_m j_m\left(\tilde{k}_0 r_1\right) + A_m h^{(1)}_m\left(\tilde{k}_0 r_1\right) = B_m h^{(2)}_m\left(\tilde{k}_1 r_1\right) + C_m h^{(1)}_m\left(\tilde{k}_1 r_1\right) \quad (2.22)$$

$$B_m h^{(2)}_m\left(\tilde{k}_1 r_2\right) + C_m h^{(1)}_m\left(\tilde{k}_1 r_2\right) = D_m j_m\left(\tilde{k}_2 r_2\right) \quad (2.23)$$

These equations can then be arranged into matrix form:

$$\begin{pmatrix} \frac{h^{(1)\prime}_m(\tilde{k}_0 r_1)}{Z_0} & \frac{h^{(2)\prime}_m(\tilde{k}_1 r_1)}{-Z_1} & \frac{h^{(1)\prime}_m(\tilde{k}_1 r_1)}{-Z_1} & 0 \\ -h^{(1)}_m\left(\tilde{k}_0 r_1\right) & h^{(2)}_m\left(\tilde{k}_1 r_1\right) & h^{(1)}_m\left(\tilde{k}_1 r_1\right) & 0 \\ 0 & \frac{h^{(2)\prime}_m(\tilde{k}_1 r_2)}{Z_1} & \frac{h^{(1)\prime}_m(\tilde{k}_1 r_2)}{Z_1} & \frac{j'_m(\tilde{k}_2 r_2)}{-Z_2} \\ 0 & h^{(2)}_m\left(\tilde{k}_1 r_2\right) & h^{(1)}_m\left(\tilde{k}_1 r_2\right) & -j_m\left(\tilde{k}_2 r_2\right) \end{pmatrix} \times \begin{pmatrix} A_m \\ B_m \\ C_m \\ D_m \end{pmatrix} = \begin{pmatrix} \frac{-\mathscr{P}_0}{Z_0}(2m+1)\mathscr{L}_m j'_m\left(\tilde{k}_0 r_1\right) \\ \mathscr{P}_0(2m+1)\mathscr{L}_m j_m\left(\tilde{k}_0 r_1\right) \\ 0 \\ 0 \end{pmatrix} \quad (2.24)$$

The coefficients can then be solved for analytically using Cramer's rule or numerically using LU decomposition.

2.1.1 Verification

By setting the outer sphere to have the same properties of the medium, or setting the two spheres to have the same properties, the problem becomes that of a single sphere of radius a which has the following solution for the plane wave case [8].

The pressure for $r > a$ is

$$p(r,\theta) = \sum_{m=0}^{\infty} \hat{A}_m P_m(\mu) h_m^{(1)}(k_{med} r) e^{-i\omega t} + \mathscr{P}_0 \sum_{m=0}^{\infty} (2m+1)(-i)^m P_m(\mu) j_m(k_{med} r) e^{-i\omega t} \quad (2.25)$$

and for $r < a$ is

$$p(r,\theta) = \sum_{m=0}^{\infty} \hat{B}_m P_m(\mu) j_m(k_{in} r) e^{-i\omega t} \quad (2.26)$$

This system of equations can be arranged to be of the form

$$\begin{pmatrix} h_m^{(1)}(k_{med}a) & -j_m(k_{sph}a) \\ \frac{1}{\rho_{med}c_{med}} h_m^{(1)\prime}(k_{med}a) & \frac{-1}{\rho_{sph}c_{sph}} j_m^{\prime}(k_{sph}a) \end{pmatrix} \begin{pmatrix} \hat{A}_m \\ \hat{B}_m \end{pmatrix} = \begin{pmatrix} -\mathscr{P}_0 (2m+1)(-i)^m j_m(k_{med}a) \\ \frac{-1}{\rho_{med}c_{med}} \mathscr{P}_0 (2m+1)(-i)^m j_m^{\prime}(k_{med}a) \end{pmatrix} \quad (2.27)$$

where k_{med}, c_{med}, ρ_{med} are properties of the surrounding medium and k_{sph}, c_{sph}, and ρ_{sph} are properties of the fluid sphere with radius a. The solution is then

$$\hat{A}_m = \mathscr{P}_0(2m+1)(-i)^m \times \frac{\left(j_m^{\prime}(k_{med}a) j_m(k_{sph}a) \rho_{sph}c_{sph} - j_m^{\prime}(k_{sph}a) n_m(k_{med}a) \rho_{med}c_{med}\right)}{-h_m^{(1)\prime}(k_{med}a) j_m(k_{sph}a) \rho_{sph}c_{sph} + j_m^{\prime}(k_{sph}a) h_m^{(1)}(k_{med}a) \rho_{med}c_{med}} \quad (2.28)$$

$$\hat{B}_m = \mathscr{P}_0(2m+1)(-i)^m \times \frac{\left(n_m^{\prime}(k_{med}a) j_m(k_{med}a) - j_m^{\prime}(k_{med}a) n_m(k_{med}a)\right) \rho_{sph}c_{sph}}{-i h_m^{(1)\prime}(k_{med}a) j_m(k_{sph}a) \rho_{sph}c_{sph} + i j_m^{\prime}(k_{sph}a) h_m^{(1)}(k_{med}a) \rho_{med}c_{med}} \quad (2.29)$$

Outer sphere matched to medium

Let the outer sphere be matched to the medium. Then $k_0 = k_1 = k_{med}$, $c_0 = c_1 = c_{med}$, $\rho_0 = \rho_1 = \rho_{med}$. Also, let $r_2 = a$, $k_2 = k_{sph}$, $c_2 = c_{sph}$,

$\rho_2 = \rho_{sph}$, and $\mathscr{L}_m = (-i)^m$ so the the problem now becomes equivalent to a plane wave incident on a single fluid sphere of radius a. For the solution to agree with that in [8], the following equations must be satisfied:

For $r > r_1$,

$$\mathscr{P}_0 \left(2m+1\right) \left(-i\right)^m j_m \left(k_{med} r\right) + A_m h_m^{(1)} \left(k_{med} r\right) \\ = \hat{A}_m h_m^{(1)}(k_{med} r) + \mathscr{P}_0 (2m+1)(-i)^m j_m(k_{med} r) \quad (2.30)$$

so

$$A_m = \hat{A}_m \quad (2.31)$$

For $a < r < r_1$,

$$B_m h_m^{(2)}(k_{med} r) + C_m h_m^{(1)}(k_{med} r) = \\ \hat{A}_m h_m^{(1)}(k_{med} r) + \mathscr{P}_0 (2m+1)(-i)^m j_m(k_{med} r) \quad (2.32)$$

The following identity is then used:

$$h_m^{(1)}(x) + h_m^{(2)}(x) = 2 j_m(x) \quad (2.33)$$

Then

$$B_m = \frac{\mathscr{P}_0 (2m+1)(-i)^m}{2} \quad (2.34)$$

and

$$C_m = B_m + \hat{A}_m \quad (2.35)$$

For $r < a$,

$$D_m P_m \left(\mu\right) j_m \left(k_{sph} r\right) = \hat{B}_m P_m \left(\mu\right) j_m \left(k_{sph} r\right) \quad (2.36)$$

so

$$D_m = \hat{B}_m \quad (2.37)$$

The solution to Eq. (2.24) for the case when the outer sphere is matched to the medium is

$$A_m = -\mathscr{P}_0(2m+1)(-i)^m \times \frac{\left(j'_m\left(k_{sph}a\right)j_m\left(k_{med}a\right)\rho_{med}c_{med} - j'_m\left(k_{med}a\right)j_m\left(k_{sph}a\right)\rho_{sph}c_{sph}\right)}{j'_m\left(k_{sph}a\right)h^{(1)}_m\left(k_{med}a\right)\rho_{med}c_{med} - h^{(1)\prime}_m\left(k_{med}a\right)j_m\left(k_{sph}a\right)\rho_{sph}c_{sph}}$$

$$B_m = \frac{\mathscr{P}_0(2m+1)(-i)^m}{2}$$

$$C_m = -\mathscr{P}_0(2m+1)(-i)^m \times \frac{\left(j'_m\left(k_{sph}a\right)h^{(2)}_m\left(k_{med}a\right)\rho_{med}c_{med} - h^{(2)\prime}_m\left(k_{med}a\right)j_m\left(k_{sph}a\right)\rho_{sph}c_{sph}\right)}{2j'_m\left(k_{sph}a\right)h^{(1)}_m\left(k_{med}a\right)\rho_{med}c_{med} - 2h^{(2)\prime}_m\left(k_{med}a\right)j_m\left(k_{sph}a\right)\rho_{sph}c_{sph}}$$

$$D_m = \mathscr{P}_0(2m+1)(-i)^m \times \frac{\left(n'_m\left(k_{med}a\right)j_m\left(k_{med}a\right) - j'_m\left(k_{med}a\right)n_m\left(k_{med}a\right)\right)\rho_{sph}c_{sph}}{ij'_m\left(k_{sph}a\right)h^{(1)}_m\left(k_{med}a\right)\rho_{med}c_{med} - ih^{(1)\prime}_m\left(k_{med}a\right)j_m\left(k_{sph}a\right)\rho_{sph}c_{sph}}$$

which satisfy Eqs. (2.30)-(2.37).

Inner and outer spheres matched

Now let the two spheres be matched to each other such that $k_1 = k_2 = k_{sph}$, $c_1 = c_2 = c_{sph}$, and $\rho_1 = \rho_2 = \rho_{sph}$. Also let $k_0 = k_{med}$, $c_0 = c_{med}$, $\rho_0 = \rho_{sph}$, and $\mathscr{L}_m = (-i)^m$. By setting r_1 to a, the problem again becomes the problem of a plane wave incident on a single sphere. The following equations must then be satisfied:

For $r > a$,

$$\mathscr{P}_0\left(2m+1\right)\left(-i\right)^m j_m\left(k_{med}r\right) + A_m h^{(1)}_m\left(k_{med}r\right) = \hat{A}_m h^{(1)}_m(k_{med}r) + \mathscr{P}_0(2m+1)(-i)^m j_m(k_{med}r) \quad (2.38)$$

so

$$A_m = \hat{A}_m \quad (2.39)$$

For $a < r < r_2$,

$$B_m h_m^{(2)}(k_{sph}r) + C_m h_m^{(1)}(k_{sph}r) = \hat{B}_m j_m(k_{sph}r) \tag{2.40}$$

so

$$B_m = C_m = \frac{\hat{B}_m}{2} \tag{2.41}$$

For $r < r_2$,

$$D_m j_m(k_{sph}r) = \hat{B}_m j_m(k_{sph}r) \tag{2.42}$$

so

$$D_m = \hat{B}_m \tag{2.43}$$

Solving Eq. (2.24) for the matched spheres case results in the following coefficients:

$$\begin{aligned} A_m = &-\mathscr{P}_0(2m+1)(-i)^m \\ &\times \frac{\left(j_m'\left(k_{sph}a\right) j_m\left(k_{med}a\right) \rho_{med}c_{med} - j_m'\left(k_{med}a\right) j_m\left(k_{sph}a\right) \rho_{sph}c_{sph}\right)}{j_m'\left(k_{sph}a\right) h_m^{(1)}\left(k_{med}a\right) \rho_{med}c_{med} - h_m^{(1)\prime}\left(k_{med}a\right) j_m\left(k_{sph}a\right) \rho_{sph}c_{sph}} \end{aligned}$$

$$\begin{aligned} B_m = &\frac{\mathscr{P}_0(2m+1)(-i)^m}{2i} \\ &\times \frac{\left(n_m'\left(k_{med}a\right) j_m\left(k_{med}a\right) - j_m'\left(k_{med}a\right) n_m\left(k_{med}a\right)\right) \rho_{sph}c_{sph}}{j_m'\left(k_{sph}a\right) h_m^{(1)}\left(k_{med}a\right) \rho_{med}c_{med} - h_m^{(1)\prime}\left(k_{med}a\right) j_m\left(k_{sph}a\right) \rho_{sph}c_{sph}} \end{aligned}$$

$$\begin{aligned} C_m = &\frac{\mathscr{P}_0(2m+1)(-i)^m}{2i} \\ &\times \frac{\left(n_m'\left(k_{med}a\right) j_m\left(k_{med}a\right) - j_m'\left(k_{med}a\right) n_m\left(k_{med}a\right)\right) \rho_{sph}c_{sph}}{j_m'\left(k_{sph}a\right) h_m^{(1)}\left(k_{med}a\right) \rho_{med}c_{med} - h_m^{(1)\prime}\left(k_{med}a\right) j_m\left(k_{sph}a\right) \rho_{sph}c_{sph}} \end{aligned}$$

$$\begin{aligned} D_m = &\mathscr{P}_0(2m+1)(-i)^m \\ &\times \frac{\left(n_m'\left(k_{med}a\right) j_m\left(k_{med}a\right) - j_m'\left(k_{med}a\right) n_m\left(k_{med}a\right)\right) \rho_{sph}c_{sph}}{i j_m'\left(k_{sph}a\right) h_m^{(1)}\left(k_{med}a\right) \rho_{med}c_{med} - i h_m^{(1)\prime}\left(k_{med}a\right) j_m\left(k_{sph}a\right) \rho_{sph}c_{sph}} \end{aligned}$$

which satisfy Eqs. (2.38)-(2.43).

2.2 Synthesizing Finite Duration Solutions from Time-Harmonic Solutions

The solution to sound scattering from two concentric fluid spheres shown above is for a time-harmonic excitation at frequency f. This time-harmonic solution has an infinite duration while the wavefront reconstruction techniques presented later on require a finite duration pulse. One can approximate the solution to finite duration pulse excitation from the time-harmonic solution using Fourier analysis. Let $v(t)$ denote the desired pulse and its Fourier transform $V(f)$, where $V(f)$ is zero for $|f|$ greater than f_{bw}. Analytically, $v(t)$ can be synthesized from its Fourier transform as

$$v(t) = \int_{-\infty}^{\infty} V(f)e^{j2\pi ft}dt \tag{2.44}$$

$$= \int_{-f_{bw}}^{f_{bw}} V(f)e^{j2\pi ft}dt \tag{2.45}$$

Computationally this would require an infinite number of frequencies to be added and is therefore not realizable. One solution to this problem is to create a T-periodic version of $v(t)$, $v((t))_T$, where

$$v((t))_T = \sum_{k=-\infty}^{\infty} v(t-kT) \tag{2.46}$$

This periodic version of $v(t)$ therefore has a Fourier series representation,

$$v((t))_T = \sum_{k=-\infty}^{\infty} V[k]e^{j2\pi kt/T} \tag{2.47}$$

where

$$V[k] = \frac{1}{T}\int_{<T>} v((t))_T e^{-j2\pi kt/T} \tag{2.48}$$

Since $v(t)$ is band-limited, $v((t))_T$ is also, so $V[k]$ is zero for $|k| > Tf_{bw}$. As a result, the sum in Eq. (2.47) is now over a finite number of frequencies

$$v((t))_T = \sum_{k=-\lfloor Tf_{bw}\rfloor}^{\lfloor Tf_{bw}\rfloor} V[k]e^{j2\pi kt/T} \tag{2.49}$$

Now let $P(r, \theta, \phi, f)$ denote the complex time-harmonic solution for a frequency f to scattering from two concentric spheres derived earlier. One can then find the approximate solution for pulse excitation using the equation

$$p((r, \theta, \phi, t))_T = \sum_{k=-\lfloor Tf_{bw} \rfloor}^{\lfloor Tf_{bw} \rfloor} P(r, \theta, \phi, -k/T) V[k] e^{j2\pi kt/T} \tag{2.50}$$

Assuming the pressure at (r, θ, ϕ) and the desired pulse are both real-valued, Eq. (2.50) simplifies to

$$\begin{aligned} p((r, \theta, \phi, t))_T = {} & V[0] P^*(r, \theta, \phi, 0) \\ & + 2 \sum_{k=1}^{\lfloor Tf_{bw} \rfloor} \operatorname{Re}\left\{ V[k] P^*(r, \theta, \phi, k/T) e^{j2\pi kt/T} \right\} \end{aligned} \tag{2.51}$$

When using this method for simulation, one must be careful when choosing the period T. Choosing T to be small reduces computation time by limiting the number of frequencies that are added in Eq. (2.51). By reducing this period, though, pulses are effectively impinging on the spheres closer together; so if the solution has a long impulse response, a new pulse will arrive before the last pulse's response died out and time-domain aliasing will occur.