

APPENDIX C: DERIVATION OF THE BIOHEAT EQUATION

For many years, there has been considerable interest in the propagation of temperature variations in the human body. This interest has been spawned by the large number of biological effects that are related to temperature changes in tissue. The earliest development was given by *Pennes* [1948] in his study of blood temperatures in the resting human forearm. In his article, Pennes provided a differential equation, the bioheat equation, that could be used to calculate temperature distributions in the human body based on thermal conduction. Pennes assumed that the medium was thermally homogeneous in order to obtain a tractable problem.

Due to modern computational power, however, the homogeneous tissue assumption is no longer required. Therefore, in this appendix, the complete bioheat equation will be derived from the basic principles of energy conservation. The derivation in the first section is similar to that provided by *Kakac and Yener* [1985] in that the substance is assumed to be incompressible and no work is being done on the fluid particles. After completing this derivation, the results are generalized to include both compressible materials as well as materials undergoing work related interactions. Finally, the effect of relaxation processes and elasticity in an isotropic medium are considered due to their importance in biological materials.

Heat Conduction in Incompressible Substance without Work Acting on System:

The first step in deriving the bioheat equation is to temporarily assume that the medium of interest is incompressible, density constant with respect to time, and that no work is being done on any of the fluid particles. Because the density is an intensive property, this means that the state of the system is uniquely determined by only one other intensive property (i.e., $p = p(T)$). As a result, the pressure p must be a function of time and space because the temperature distribution should also be a function of time and space. Hence, in order not to have any work acting on the system, the particle velocity \vec{u} and the particle displacement $\vec{\xi}_d$ must also be zero. Hence, the energy conservation equation given in Appendix A simplifies to yield

$$\begin{aligned}
\frac{\partial}{\partial t} \iiint_{V^*} \rho e dV &= \iiint_{V^*} \dot{q}_i dV - \iint_{S^*} \dot{\vec{q}} \cdot \hat{n} dS \\
\Rightarrow \iiint_{V^*} \left(\rho \frac{De}{Dt} - \dot{q}_i + \nabla \cdot \dot{\vec{q}} \right) dV &= 0 \\
\Rightarrow \iiint_{V^*} \left(\rho \left(\frac{\partial e}{\partial t} + (\vec{u} \cdot \nabla) e \right) - \dot{q}_i + \nabla \cdot \dot{\vec{q}} \right) dV &= 0 \\
\Rightarrow \rho \frac{\partial e}{\partial t} - \dot{q}_i + \nabla \cdot \dot{\vec{q}} &= 0,
\end{aligned} \tag{C.1}$$

where e is the thermodynamic internal energy, ρ is the density, \dot{q}_i is the contribution of any heat sources/sinks in the medium, and $\dot{\vec{q}}$ is the heat flow due to thermal conduction.

In order to complete the derivation of the bioheat equation for this special case, use is made of Fourier's Law of heat conduction (i.e., $\dot{\vec{q}} = -\kappa_t \nabla T$ where κ_t is the thermal conductivity of the medium, and T is the temperature) and the thermodynamic relationship $e = e(T)$ resulting from the incompressibility of the medium. Applying basic principles of differential equations then yields

$$\begin{aligned}
de &= \frac{\partial e}{\partial T} dT = c_V dT \\
\Rightarrow \frac{\partial e}{\partial t} &= c_V \frac{\partial T}{\partial t},
\end{aligned} \tag{C.2}$$

where c_V is the specific heat at constant volume. As a result, Equation (C.1) becomes

$$\rho c_V \frac{\partial T}{\partial t} = \dot{q}_i + \nabla \cdot \kappa_t \nabla T \tag{C.3}$$

which is the general heat conduction equation. The bioheat equation is formed by simply separating the \dot{q}_i into a heat source term and a blood perfusion term as shown in Equation (C.4) [Pennes, 1948].

$$\rho c_V \frac{\partial T}{\partial t} = \dot{q}_{blood} + \dot{q}_{source} + \nabla \cdot \kappa_t \nabla T \tag{C.4}$$

Heat Conduction in Compressible Substance without Work Acting on System:

The previous analysis was restricted to incompressible substances. However, in general, the material will be compressible, so the question remains as to whether the bioheat equation would still apply. Once again, \vec{u} must be zero in order to avoid work done by shearing forces. As a result, the only change in the previous analysis pertains to writing the change in internal

energy e as a function of a change in the temperature T in Equation (C.2). Therefore, Equation (C.2) becomes

$$\begin{aligned} de &= \left(\frac{\partial e}{\partial \rho} \right)_T d\rho + c_v dT \\ \Rightarrow \frac{\partial e}{\partial t} &= \left(\frac{\partial e}{\partial \rho} \right)_T \frac{\partial \rho}{\partial t} + c_v \frac{\partial T}{\partial t}. \end{aligned} \quad (C.5)$$

Furthermore, applying the conservation of mass [Pierce, 1991] yields

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \bar{u}) = 0 \Rightarrow \frac{\partial e}{\partial t} = c_v \frac{\partial T}{\partial t} \quad (C.6)$$

because \bar{u} is zero. Therefore, the bioheat equation would be exactly the same for a compressible material as for an incompressible material.

Heat Conduction in Compressible Substance with Work Acting on System:

For many heat conduction problems, the analysis up to this point would suffice because there is no mechanical work acting on the fluid particles. For example, heating due to electromagnetic exposure as was done by *Paulsen et al.* [2001] can be accurately characterized. Unfortunately, when dealing with heating due to ultrasound, the mechanical interactions of the fluid particles can no longer be ignored (i.e., $\bar{u} \neq 0$). Therefore, it is necessary to return to the energy conservation equation to determine if another heat conduction equation could be found for this case by separating the thermal and the mechanical effects. During this derivation, all thermodynamic processes will be assumed to be in quasi-equilibrium and elastic forces will be assumed negligible. Relaxation and elastic effects will be considered after this initial analysis is complete.

The conservation of energy equation for a viscous medium neglecting elasticity is given by

$$\frac{\partial}{\partial t} \iiint_{V^*} \left(\frac{1}{2} \rho \bar{u} \cdot \bar{u} + \rho e \right) dV = \iiint_{V^*} \dot{q}_i dV - \iint_{S^*} \dot{\bar{q}} \cdot \hat{n} dS + \iint_{S^*} \sigma_n \bar{u} \cdot \hat{n} dS + \iint_{S^*} \mu \sum_{i,j=1}^3 \phi_{ij} u_i \hat{e}_j \cdot \hat{n} dS \quad (C.7)$$

where σ_n is the average of the normal components of the stress tensor and μ is the shear viscosity [Pierce, 1991]. For a simple viscous medium, σ_n is just the negative of the pressure p [Pierce, 1991]. The first step in separating the thermal and mechanical effects is to make use of the thermodynamic identity [Cengel and Boles, 1994]

$$de = c_v dT + \left[T \left(\frac{\partial p}{\partial T} \right)_\rho - p \right] d\rho^{-1}. \quad (\text{C.8})$$

Substituting this value for the internal energy into Equation (C.7) yields

$$\begin{aligned} & \iiint_{V^*} \left(\frac{1}{2} \rho \frac{D}{Dt} (\bar{u} \cdot \bar{u}) + \rho c_v \frac{DT}{Dt} + \rho \left[T \left(\frac{\partial p}{\partial T} \right)_\rho - p \right] \frac{D\rho^{-1}}{Dt} \right) dV \\ &= \iiint_{V^*} \dot{q}_i dV - \iint_{S^*} \dot{\bar{q}} \cdot \hat{n} dS - \iint_{S^*} p \bar{u} \cdot \hat{n} dS + \iint_{S^*} \mu \sum_{i,j=1}^3 \phi_{ij} u_i \hat{e}_j \cdot \hat{n} dS. \end{aligned} \quad (\text{C.9})$$

Now, transform this equation into an equation involving small perturbations about the ambient state by assuming

$$\begin{aligned} p &= p_o + p' \\ T &= T_c + T' \\ \rho &= \rho_c + \rho' \\ \bar{u} &= \bar{u}'. \end{aligned} \quad (\text{C.10})$$

These values can be substituted into Equation (C.9) to yield

$$\begin{aligned} & \iiint_{V^*} \left(\frac{\rho}{2} \frac{D}{Dt} (\bar{u} \cdot \bar{u}) + \rho c_v \frac{DT'}{Dt} - \left[T' \left(\frac{\partial p}{\partial T} \right)_\rho - p' \right] \frac{1}{\rho} \frac{D\rho}{Dt} - \left[T_c \left(\frac{\partial p}{\partial T} \right)_\rho - p_o \right] \frac{1}{\rho} \frac{D\rho}{Dt} \right) dV \\ &= \left(\iiint_{V^*} (\dot{q}_i + \nabla \cdot \kappa_i \nabla T') dV + \iint_{S^*} \mu \sum_{i,j=1}^3 \phi_{ij} u_i \hat{e}_j \cdot \hat{n} dS \right) - \iiint_{V^*} \nabla \cdot ((p_o + p') \bar{u}) dV \\ &= \left(\iiint_{V^*} (\dot{q}_i + \nabla \cdot \kappa_i \nabla T' - \nabla \cdot (p' \bar{u})) dV + \iint_{S^*} \mu \sum_{i,j=1}^3 \phi_{ij} u_i \hat{e}_j \cdot \hat{n} dS \right) - \iiint_{V^*} p_o \nabla \cdot \bar{u} dV \\ &= \left(\iiint_{V^*} (\dot{q}_i + \nabla \cdot \kappa_i \nabla T' - \nabla \cdot (p' \bar{u})) dV + \iint_{S^*} \mu \sum_{i,j=1}^3 \phi_{ij} u_i \hat{e}_j \cdot \hat{n} dS \right) + \iiint_{V^*} \frac{p_o}{\rho} \frac{D\rho}{Dt} dV, \end{aligned} \quad (\text{C.11})$$

where the conservation of mass equation [Pierce, 1991] has been manipulated to yield

$$\begin{aligned} 0 &= \nabla \cdot (\rho \bar{u}) + \frac{\partial \rho}{\partial t} \Rightarrow -\frac{\partial \rho}{\partial t} = \bar{u} \cdot \nabla \rho + \rho \nabla \cdot \bar{u} \\ &\Rightarrow \nabla \cdot \bar{u} = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial t} + \bar{u} \cdot \nabla \rho \right) \\ &\Rightarrow \nabla \cdot \bar{u} = -\frac{1}{\rho} \frac{D\rho}{Dt}. \end{aligned} \quad (\text{C.12})$$

Therefore, Equation (C.11) becomes

$$\begin{aligned}
& \iiint_{V^*} \left(\frac{1}{2} \rho \frac{D}{Dt} (\bar{u} \cdot \bar{u}) + \rho c_V \frac{DT'}{Dt} - \left[T' \left(\frac{\partial p}{\partial T} \right)_\rho - p' \right] \frac{1}{\rho} \frac{D\rho}{Dt} - \frac{T_c}{\rho} \left(\frac{\partial p}{\partial T} \right)_\rho \frac{D\rho}{Dt} \right) dV \\
& = \left(\iiint_{V^*} (\dot{q}_i + \nabla \cdot \kappa_i \nabla T' - \nabla \cdot (p' \bar{u})) dV + \iint_{S^*} \mu \sum_{i,j=1}^3 \phi_{ij} u_i \hat{e}_j \cdot \hat{n} dS \right).
\end{aligned} \tag{C.13}$$

Applying

$$\begin{aligned}
dp & = \left(\frac{\partial p}{\partial T} \right)_\rho dT + \left(\frac{\partial p}{\partial \rho} \right)_T d\rho \\
\Rightarrow p' & = \left(\frac{\partial p}{\partial T} \right)_\rho T' + c_T^2 \rho'
\end{aligned} \tag{C.14}$$

then yields

$$\begin{aligned}
& \iiint_{V^*} \left(\frac{1}{2} \rho \frac{D}{Dt} (\bar{u} \cdot \bar{u}) + \rho c_V \frac{DT'}{Dt} + \frac{c_T^2 \rho'}{\rho} \frac{D\rho}{Dt} \right) dV - \iiint_{V^*} \left(\frac{T_c}{\rho} \left(\frac{\partial p}{\partial T} \right)_\rho \frac{D\rho}{Dt} \right) dV \\
& = \left(\iiint_{V^*} \left(\dot{q}_i + \nabla \cdot \kappa_i \nabla T' - \nabla \cdot \left[\bar{u} \left[\left(\frac{\partial p}{\partial T} \right)_\rho T' + c_T^2 \rho' \right] \right] \right) dV + \iint_{S^*} \mu \sum_{i,j=1}^3 \phi_{ij} u_i \hat{e}_j \cdot \hat{n} dS \right),
\end{aligned} \tag{C.15}$$

where c_T is speed of sound assuming isothermal propagation. Also,

$$\left(\frac{\partial p}{\partial T} \right)_\rho = - \left(\frac{\partial p}{\partial \rho} \right)_T \left(\frac{\partial \rho}{\partial T} \right)_p = \rho c_T^2 \beta_{therm}, \tag{C.16}$$

where β_{therm} is the coefficient of thermal expansion [Pierce, 1991] leaving

$$\begin{aligned}
& \iiint_{V^*} \left(\frac{1}{2} \rho \frac{D}{Dt} (\bar{u} \cdot \bar{u}) + \rho c_V \frac{DT'}{Dt} + \frac{c_T^2 \rho'}{\rho} \frac{D\rho}{Dt} \right) dV - \iiint_{V^*} \left([T_c c_T^2 \beta_{therm}] \frac{D\rho}{Dt} \right) dV \\
& = \left(\iiint_{V^*} (\dot{q}_i + \nabla \cdot \kappa_i \nabla T' - \nabla \cdot (\bar{u} [\rho c_T^2 \beta_{therm} T' + c_T^2 \rho'])) dV + \iint_{S^*} \mu \sum_{i,j=1}^3 \phi_{ij} u_i \hat{e}_j \cdot \hat{n} dS \right).
\end{aligned} \tag{C.17}$$

Because only the first-order perturbations in the energy are of interest, only the lowest order terms are retained. This would be terms containing only T' or terms that are second order in the other variables. Therefore, terms like $\bar{u} T'$ would be eliminated because it is not of the lowest order. This yields

$$\begin{aligned}
& \iiint_{V^*} \left(\frac{1}{2} \rho_c \frac{\partial}{\partial t} (\bar{u} \cdot \bar{u}) + \rho_c c_V \frac{\partial T'}{\partial t} + \frac{c_T^2 \rho'}{\rho_c} \frac{\partial \rho'}{\partial t} \right) dV - \iiint_{V^*} \left(T_c c_T^2 \beta_{therm} \frac{\partial \rho'}{\partial t} \right) dV \\
& = \left(\iiint_{V^*} (\dot{q}_i + \nabla \cdot \kappa_i \nabla T' - \nabla \cdot (\bar{u} c_T^2 \rho')) dV + \iint_{S^*} \mu \sum_{i,j=1}^3 \phi_{ij} u_i \hat{e}_j \cdot \hat{n} dS \right).
\end{aligned} \tag{C.18}$$

Now, if the ratio of the specific heats is very close to one, as is the case for biological tissue, Equation (C.18) becomes

$$\left(\frac{1}{2} \rho_c \frac{\partial}{\partial t} (\bar{\mathbf{u}} \cdot \bar{\mathbf{u}}) + \nabla \cdot (\bar{\mathbf{u}} p') + \frac{1}{2 \rho_c c_T^2} \frac{\partial p'^2}{\partial t} - \nabla \cdot \sum_{i,j=1}^3 \mu \phi_{ij} u_i \hat{\mathbf{e}}_j \right) - T_c \beta_{therm} \frac{\partial p'}{\partial t} = \dot{q}_i + \nabla \cdot \kappa_i \nabla T' - \rho_c c_V \frac{\partial T'}{\partial t}. \quad (\text{C.19})$$

Furthermore, the $T_c \beta_{therm}$ term can also be neglected if its order is compared to the pressure squared term. $T_c \beta_{therm} \sim 300 \cdot 8.8 \cdot 10^{-5} = 0.0264$, but the pressure squared term is of order $\frac{\omega p'}{2 \rho_c c_T^2} \sim \frac{\pi 10^6 \cdot 500 \cdot 10^3}{1000 \cdot 1487^2} = 710$ when the frequency is in the megahertz range and the pressure is at least 500 kPa. Hence, Equation (C.19) can be written as

$$\frac{\partial w}{\partial t} + \nabla \cdot I' = \dot{q}_i + \nabla \cdot \kappa_i \nabla T' - \rho_c c_V \frac{\partial T'}{\partial t}, \quad (\text{C.20})$$

where

$$w = \frac{\rho_o (\bar{\mathbf{u}} \cdot \bar{\mathbf{u}})}{2} + \frac{p'^2}{2 \rho_c c_T^2} \quad (\text{C.21})$$

$$I' = \bar{\mathbf{u}} p' - \sum_{i,j=1}^3 \mu \phi_{ij} u_i \hat{\mathbf{e}}_j.$$

Equation (C.20) is very close to the previously derived bioheat equation with the left-hand side acting as an additional heat source. More will be said about the left-hand side after elastic and relaxation effects have been included.

Heat Conduction in an Elastic System with Relaxation Processes

In the previous section, the expressions for heat conduction in a simple viscous medium were developed. However, biological tissues are affected by relaxation processes and elastic forces that affect the conservation of energy in the thermal-mechanical interactions. As a result, the complete energy conservation equation provided in Appendix A will need to be considered. For convenience, the equation is restated below:

$$\frac{\partial}{\partial t} \iiint_{V^*} \left(\frac{1}{2} \rho \bar{\mathbf{u}} \cdot \bar{\mathbf{u}} + \rho e + \frac{1}{2} \sum_{i,j=1}^3 (\lambda_L \theta_d \delta_{ij} \xi_{d_i} + 2 \mu_L S_{ij} \xi_{d_i}) \right) dV \quad (\text{C.22})$$

$$= \iiint_{V^*} \dot{q}_i dV - \iint_{S^*} \dot{\mathbf{q}} \cdot \hat{\mathbf{n}} dS + \iint_{S^*} \sum_{i,j=1}^3 \sigma_{ij} u_i \hat{\mathbf{e}}_j \cdot \hat{\mathbf{n}} dS.$$

In this equation, \dot{q}_i is the heat generated within the particle, $\dot{\bar{q}}$ is the heat flow across the boundary of the particle, and σ is a tensor representing the external forces acting on the fluid particle. For an isotropic viscous relaxing elastic medium, σ is given by [O'Brien, 2000]

$$\sigma_{ij} = \lambda_L \theta_d \delta_{ij} + 2\mu_L S_{ij} + \mu_B \dot{\theta}_d \delta_{ij} + 2\mu \dot{S}_{ij} \quad (C.23)$$

whose parameters are the Lamé constants λ_L and μ_L , the bulk and shear viscosity μ_B and μ , the dilatation θ_d , and the tensor strain S . The other parameters in Equation (C.22) are defined as the outward unit normal for the particle \hat{n} , the velocity \vec{u} , the displacement $\vec{\xi}_d$, and the density ρ .

Now separate out the temperature from the internal energy similar to the previous cases. A discussion of internal energy with relaxation effects included can be found in Pierce [1991]; however, for this derivation the following relation will be sufficient:

$$de = Tds - pd\rho^{-1} - \sum_{\nu} A_{\nu} dT_{\nu}. \quad (C.24)$$

In this equation, s is the entropy, and the summation captures the energies of the vibrational motions of the molecules. Substituting this equation into the energy conservation equation yields

$$\begin{aligned} & \iiint_{V^*} \left(\frac{1}{2} \frac{D}{Dt} \sum_{i,j=1}^3 (\lambda_L \theta_d \delta_{ij} \xi_{di} + 2\mu_L S_{ij} \xi_{di}) + \frac{\rho}{2} \frac{D(\vec{u} \cdot \vec{u})}{Dt} + \rho T \frac{Ds}{Dt} + \frac{p}{\rho} \frac{D\rho}{Dt} - \sum_{\nu} \rho A_{\nu} \frac{DT_{\nu}}{Dt} \right) dV \\ & = \iiint_{V^*} \dot{q}_i dV - \iint_{S^*} \dot{\bar{q}} \cdot \hat{n} dS + \iint_{S^*} \sum_{i,j=1}^3 \sigma_{ij} u_i \hat{e}_j \cdot \hat{n} dS. \end{aligned} \quad (C.25)$$

Now decompose the entropy into two distinct contributions s_{fr} and s_{ν} [Pierce, 1991]:

$$s = s_{fr} + \sum_{\nu} s_{\nu}. \quad (C.26)$$

In this equation, s_{fr} corresponds to the entropy associated with translational and rotational motions of the fluid particles, and s_{ν} captures the entropy associated with the much smaller vibrational motions. The subscript fr refers to a frozen state where molecules are not allowed to vibrate, and the subscript ν refers to the different possible vibrational modes for the molecules. Then from thermodynamics [Pierce, 1991]

$$ds_{\nu} = \frac{1}{T_{\nu}} de_{\nu} = \frac{c_{\nu\nu}}{T_{\nu}} dT_{\nu} \quad (C.27)$$

and [Cengel and Boles, 1994]

$$\begin{aligned}
ds_{fr} &= \left(\frac{\partial s_{fr}}{\partial T} \right)_\rho dT + \left(\frac{\partial s_{fr}}{\partial \rho} \right)_T d\rho \\
\Rightarrow ds_{fr} &= \frac{c_{vfr}}{T} dT + \left(\frac{\partial s_{fr}}{\partial p} \right)_T \left(\frac{\partial p}{\partial \rho} \right)_T d\rho \\
\Rightarrow ds_{fr} &= \frac{c_{vfr}}{T} dT + \left(\frac{\partial s_{fr}}{\partial p} \right)_T c_T^2 d\rho,
\end{aligned} \tag{C.28}$$

where the thermodynamic quantities are those appropriate for the frozen state. Applying the Maxwell relation [Cengel and Boles, 1994]

$$\left(\frac{\partial s_{fr}}{\partial p} \right)_T = - \left(\frac{\partial \rho^{-1}}{\partial T} \right)_p \tag{C.29}$$

then yields

$$\begin{aligned}
ds_{fr} &= \frac{c_{vfr}}{T} dT - \left(\frac{\partial \rho^{-1}}{\partial T} \right)_p c_T^2 d\rho \\
\Rightarrow ds_{fr} &= \frac{c_{vfr}}{T} dT - \frac{\beta_{therm}}{\rho} c_T^2 d\rho.
\end{aligned} \tag{C.30}$$

Substituting Equations (C.26), (C.27), and (C.30) into Equation (C.25) then yields

$$\begin{aligned}
& \iiint_{V^*} \left(\frac{1}{2} \sum_{i,j=1}^3 \frac{D}{Dt} (\lambda_L \theta_d \delta_{ij} \xi_{di} + 2\mu_L S_{ij} \xi_{di}) + \frac{\rho}{2} \frac{D(\bar{u} \cdot \bar{u})}{Dt} + \rho c_{vfr} \frac{DT}{Dt} \right. \\
& \left. + \left(\frac{p}{\rho} - T\beta_{therm} c_T^2 \right) \frac{D\rho}{Dt} + \sum_v \frac{\rho T c_{vv}}{T_v} \frac{DT_v}{Dt} - \sum_v \rho A_v \frac{DT_v}{Dt} \right) dV \\
& = \iiint_{V^*} \dot{q}_i dV - \iint_{S^*} \dot{\bar{q}} \cdot \hat{n} dS + \iint_{S^*} \sum_{i,j=1}^3 \sigma_{ij} u_i \hat{e}_j \cdot \hat{n} dS.
\end{aligned} \tag{C.31}$$

Equation (C.31) can be further simplified by the relation [Pierce, 1991]

$$\sum_v \rho A_v \frac{DT_v}{Dt} = \sum_v \rho c_{vv} \left(\frac{T}{T_v} - 1 \right) \frac{DT_v}{Dt} \tag{C.32}$$

to yield

$$\begin{aligned}
& \iiint_{V^*} \left(\frac{1}{2} \sum_{i,j=1}^3 \frac{D}{Dt} (\lambda_L \theta_d \delta_{ij} \xi_{di} + 2\mu_L S_{ij} \xi_{di}) + \frac{\rho}{2} \frac{D(\bar{u} \cdot \bar{u})}{Dt} + \rho c_{vfr} \frac{DT}{Dt} \right. \\
& \left. + \left(\frac{p}{\rho} - T\beta_{therm} c_T^2 \right) \frac{D\rho}{Dt} + \sum_v \rho c_{vv} \frac{DT_v}{Dt} \right) dV \\
& = \iiint_{V^*} \dot{q}_i dV - \iint_{S^*} \dot{\bar{q}} \cdot \hat{n} dS + \iint_{S^*} \sum_{i,j=1}^3 \sigma_{ij} u_i \hat{e}_j \cdot \hat{n} dS.
\end{aligned} \tag{C.33}$$

At this point, Equation (C.33) can be linearized similar to how Equation (C.9) was linearized in the preceding section. Keeping only the first-order terms as was explained before then yields

$$\begin{aligned} & \iiint_{V^*} \left(\frac{1}{2} \sum_{i,j=1}^3 \frac{\partial}{\partial t} (\lambda_L \theta_d \delta_{ij} \xi_{di} + 2\mu_L S_{ij} \xi_{di}) + \frac{\rho_c}{2} \frac{\partial(\bar{u} \cdot \bar{u})}{\partial t} + \frac{c_T^2 \rho'}{\rho_c} \frac{\partial \rho'}{\partial t} - \nabla \cdot \left(\sum_{i,j=1}^3 \sigma_{ij} u_i \hat{e}_j \right) \right) dV \\ & = \iiint_{V^*} \left(\dot{q}_i + \nabla \cdot (\kappa_t \nabla T') - \rho_c \frac{\partial}{\partial t} \left(c_{Vfr} T' + \sum_v c_{vv} T'_v \right) + T_c \beta_{therm} c_T^2 \frac{\partial \rho'}{\partial t} \right) dV. \end{aligned} \quad (C.34)$$

Now define an effective temperature T'_{eff} and effective specific heat c_{Veff} such that

$c_{Vfr} T' + \sum_v c_{vv} T'_v \rightarrow c_{Veff} T'_{eff}$ and assume that $T' \cong T'_{eff}$ because the vibrational energies of the molecules are much smaller than the translational and rotational energies. If the ratio of the specific heats is also very close to one, then

$$\frac{\partial w}{\partial t} + \nabla \cdot I' = \dot{q}_i + \nabla \cdot (\kappa_t \nabla T'_{eff}) - \rho_c c_{Veff} \frac{\partial T'_{eff}}{\partial t} \quad (C.35)$$

where in this case

$$\begin{aligned} w &= \sum_{i,j=1}^3 \frac{(\lambda_L \theta_d \delta_{ij} \xi_{di} + 2\mu_L S_{ij} \xi_{di})}{2} + \frac{\rho_c (\bar{u} \cdot \bar{u})}{2} + \frac{p^2}{2\rho_c c_T^2} \\ I' &= - \sum_{i,j=1}^3 \sigma_{ij} u_i \hat{e}_j. \end{aligned} \quad (C.36)$$

Once again the w and I' terms are effectively acting as an additional heat source for the bioheat equation.

Now the rigorous derivation of the bioheat equation can be compared to previous work in this area. Often, other researchers began by assuming that the heat source term in the bioheat equation could be written as the time derivative of the acoustical energy plus the divergence of the acoustical intensity just as was found in this derivation [Goueygou *et al.*, 1999; Nyborg, 1981; Cavicchi and O'Brien, 1984; Haken *et al.*, 1992; Fujii *et al.*, 1999]. Furthermore, the conclusions derived by these researchers have been experimentally verified by many people over the years and often are the basis for measuring the absorption coefficient of biological tissue [Fry and Fry, 1954a; Fry and Fry, 1954b; Parker, 1983; Parker, 1985]. As a result, at this point the previous work in the field will be referenced to rewrite Equations (C.36) and (C.20) as

$$\rho_c c_V \frac{\partial T'}{\partial t} = \dot{q}_i + \nabla \cdot (\kappa_t \nabla T') + \sum_{i=0}^{\infty} \alpha_{loc}(\omega_i) I'(\omega_i), \quad (\text{C.37})$$

where α_{loc} is the absorption coefficient of the tissue and I' the intensity of the acoustical disturbance at frequency ω_i . Equation (C.37) also requires periodicity of the ultrasound source as is true for most medical ultrasound applications.