CHAPTER 4

THEORETICAL ANALYSIS OF PROPOSED INDICES

As was previously shown in Chapter 1, linear changes in the applied voltage across the transducer should produce linear changes in the acoustic pressure at the focus of the transducer provided that the nonlinear effects associated with acoustic propagation do not significantly alter the wave before it reaches the focus. Once the nonlinear effects become significant, linear extrapolation is no longer valid. In order to determine when nonlinear effects can no longer be ignored, many people have proposed indicators that could potentially be used to monitor the level of acoustical nonlinearity in a propagating wave.

The indices can be grouped into two different categories, absolute indicators of nonlinearity and relative indicators of nonlinearity. The relative indicators incorporate low voltage measurements of the same transducer into the analysis, whereas the absolute indicators only require information from the present voltage setting. In this chapter, each of the indicators are introduced along with their theoretical basis, a discussion on how they will be determined, and an evaluation of their respective advantages and disadvantages.

4.1 Absolute Indicators of Nonlinearity

The first type of indicators that will be discussed are the absolute indicators of nonlinearity. These are parameters that could ideally assess the amount of nonlinear distortion present in a wave by only considering the information contained in the current waveform. For our problem of determining a reliable indicator of nonlinearity in order to find the range over which linear extrapolation would be valid, an absolute indicator would be preferred over a relative indicator because fewer measurements would be required. The absolute indicators of nonlinearity that will be discussed in this work are the asymmetric ratio, Ostrovskii/Sutin's propagation parameter s_s, Bacon's acoustic propagation parameter s_m, the field sigma s_z, the second harmonic ratio, and the absolute spectral index. Each of these indices will be defined in the subsequent sections.

4.1.1 Asymmetric ratio

After completing the analysis presented in Chapter 2 on asymmetric distortion, it would seem that a logical choice for an indicator should reflect the amount of asymmetry in the waveform. One quantity that does this is the asymmetric ratio given by

$$P_{asym} = \frac{p_c}{p_r} \tag{4.1}$$

where p_c is the peak compressional pressure, and p_r is the peak rarefractional pressure. Clearly, this indicator of nonlinearity can be directly obtained from the measured data. Simply take the ratio of the maximum and minimum values of the waveform.

The simple relationship of the asymmetric ratio to the measured data is its principle advantage. Another possible advantage would be its theoretical relationship to the error in linear extrapolation. As will be shown in the next section, the asymmetric ratio can be used to obtain s_s, which in turn is directly related to the extrapolation error. Because these relationships are all derived assuming Ostrovskii/Sutin's method as described in Chapter 2 is accurate, the exact relationships will not be derived until Section 4.1.2. Also, if Ostrovskii/Sutin's theory were not valid, then the asymmetric ratio's relationship to the error would no longer be valid.

As far as disadvantage, the only one that will be discussed in this thesis is its sensitivity to the initial symmetry in the waveform. If the waveform generated by the source were highly asymmetric, then the asymmetric ratio would no longer reflect the asymmetry generated by nonlinear propagation. For example, if the asymmetric ratio at the source was greater than one, then this indicator would reflect more nonlinearity then was actually present unnecessarily reducing the range over which linear extrapolation could be performed. Likewise, if the asymmetric ratio at the source was less than one, the indicator would show less nonlinearity then was actually present exaggerating the true range of linear extrapolation. In either case, the asymmetric ratio would fail in its intended purpose.

4.1.2 Ostrovskii/Sutin's propagation parameter s_s

The amount of asymmetric distortion in a waveform can also be described by Ostrovskii/Sutin's propagation parameter s_s. The larger the value of s_s, the more nonlinear the waveform. Because s_s was already derived in Chapter 2, we shall only elaborate on the connection between s_s and errors in the linearly extrapolated waveform in this section. In Equation (2.8), s_s was defined as

$$s_s = \frac{p_o w F b}{r c^3} \ln\left(\frac{F}{R_o}\right) \tag{4.2}$$

Before proceeding with our evaluation of s_s, we need to discuss how s_s will be determined. Ideally, s_s could be found from Equation (4.2). Unfortunately, the initial source pressure, p_o, is not known because the measurement is to be made at the focus. Therefore, it is necessary to modify Equation (4.2) to remove the dependence on p_o. Recall that the peak compressional and rarefractional pressures according to the Ostrovskii/Sutin method are given by Equation (2.12) and repeated in Equation (4.3) for convenience:

$$\begin{aligned} p_c &= \frac{a \cdot \sin(a)}{p} \frac{2 p_o F w}{c} \frac{1}{1 - s_s} \\ p_r &= \frac{a \cdot \sin(a)}{p} \frac{2 p_o F w}{c} \frac{1}{1 + s_s} \end{aligned} \tag{4.3}$$

Notice that the dependence on p_o can be removed by substituting the value of s_s given by Equation (4.2) into one of the pressure equations given by (4.3) and then solving for p_o. Since the rarefractional pressure will be effected less by nonlinear absorption, the second equation will be selected when solving for p_o.

$$p_r = \frac{\left(\frac{a \cdot \sin(a)}{p} \frac{2Fw}{c}\right) p_o}{1 + \left(\frac{wFb}{rc^3} \ln\left(\frac{F}{R_o}\right)\right) p_o} \Rightarrow p_o = \frac{p_r}{\left(\frac{a \cdot \sin(a)}{p} \frac{2Fw}{c}\right) - \left(\frac{wFb}{rc^3} \ln\left(\frac{F}{R_o}\right)\right) p_r} \tag{4.4}$$

Substituting this value for p_o back into Equation (4.2) yields

$$s_s = \frac{p_r\, b \ln\left(\frac{F}{R_o}\right)}{2\, r c^2 \left(\frac{a \cdot \sin(a)}{p}\right) - \left(p_r\, b \ln\left(\frac{F}{R_o}\right)\right)} \tag{4.5}$$

Notice that Equation (4.5) depends on the density r, sound speed c, and nonlinearity b, the peak rarefractional pressure at the focus p_r, the focal length of the transducer F, and the boundary location of the two regions of propagation R_o. Except for R_o, all of these parameters are easy to determine. There is some uncertainty in the value of R_o because it is difficult to define quantitatively, as was discussed in Chapter 2. For this thesis, the value of R_o selected for the purpose of evaluating s_s was the r_f defined in Chapter 2. Of course a different value of R_o may yield different performance.

Another method for determining s_s that avoids the problem of uncertainty in R_o is based on the asymmetric ratio defined in Equation (4.1) in the previous section. If the values for p_c and p_r found by the Ostrovskii/Sutin method given in Equation (4.3) are substituted into this ratio, then the following simple relation for determining s_s presents itself:

$$s_s = \frac{P_{asym} - 1}{P_{asym} + 1} \tag{4.6}$$

Notice that using this equation s_s, can be determined directly from a measurement without making any assumptions on the values of the other parameters. Because we know s_s can be determined from the pressure waveform at the focus, we can now perform a qualitative evaluation of its use as an indicator for nonlinearity.

The first step in evaluating s_s is to determine how it compares to our ability to perform linear extrapolation. For the purpose of this analysis, we shall assume that the Ostrovskii/Sutin method for determining the pressure waveform at the focus of a transducer is sufficiently accurate. Notice that in each of the equations given in (4.3), the pressure consists of a linear portion divided by $1 \pm s_s$. Therefore, the error between the "true" values of peak compressional p_c and peak rarefractional pressures p_r, as predicted by Ostrovskii/Sutin, and our linearly extrapolated value should vary with σ_s as

$$\%Error\_p_{r,c} = 100 \cdot \left| \left(1 - \frac{1}{1 \pm s_s} \right) (1 \pm s_s) \right| = 100 \cdot s_s \tag{4.7}$$

Likewise, the error in the peak average pressure (i.e., $p_{avg} = (p_c + p_r)/2$) should vary with s_s as

$$\%Error\_p_{avg} = 100 \cdot \left| \frac{1 - \frac{1}{2}\left(\frac{1}{1 - s_s} + \frac{1}{1 + s_s} \right)}{\frac{1}{2}\left(\frac{1}{1 - s_s} + \frac{1}{1 + s_s} \right)} \right| = 100 \cdot s_s^2 \tag{4.8}$$

As an aside, the error in the peak average pressure would be the same as the error in the peak-peak pressure if this later parameter were of importance for the measurement. From Equations (4.7) and (4.8), it is clear that "ideally" there is a simple relationship between the value of s_s and the expected errors in the linear extrapolation. However, this simple relationship may degrade if the Ostrovskii/Sutin method is not sufficiently accurate.

At this point, we can make some general observations on the advantages and disadvantages of s_s as an indicator of nonlinearity. The main advantage of s_s is that it "ideally" has a simple relationship to the expected error in the linear extrapolation. It also can be determined directly from a simple measurement as given in Equation (4.6). However, it also suffers from several disadvantages. Namely, it is only quantitatively meaningful if Ostrovskii/Sutin's theory is accurate. Once their theory breaks down, s_s loses its significance. Another disadvantage is that if s_s is determined by Equation (4.6) it may be biased by asymmetries present in the source waveform as was discussed in the previous section. Using Equation (4.5) to determine σ_s avoids this biasing; however, as was mentioned before, the R_o in this equation is difficult to determine analytically.

4.1.3 Bacon's propagation parameter s_m

The next absolute nonlinear indicator that will be discussed is Bacon's acoustic propagation parameter s_m [*Bacon*, 1984]. The goal of Bacon's analysis was very similar to our present analysis in that he wanted to determine the amount of nonlinearity in a converging sound wave based on a pressure measurement at the focus. Bacon bases his work on the analysis of a planar transducer done Fenlon and Kesner in 1976 who in turn based their work on equations derived by Kuznetsov in 1971 [*Bacon*, 1984]. In their work, Fenlon and Kesner determined the fields produced along the beam axis of a planer source at the origin assuming a Gaussian distribution across its surface and neglecting the effect of variations in the off-axis fields [*Bacon*, 1984]. Bacon then applied these ideas to a focused source by treating the focus as a virtual planar Gaussian radiator and then back propagating the waves to the transducer surface using the expressions developed by Fenlon and Kesner in 1976 [*Bacon*, 1984]. Before proceeding with Bacon's analysis, it is important to emphasis that by neglecting the off-axis field variations, Bacon has removed all asymmetric distortion from his theory. This approximation places a severe limitation on the applicability of Bacon's final expressions for the field.

Using these ideas, and assuming sinusoidal propagation, Bacon showed that the pressure field along the beam axis of the transducer after appropriate substitutions could be described by

$$p(r,t) = \frac{p_o}{\sqrt{1+R^2}} \sin\left(wt - \frac{wr}{c} + \frac{s_o p \sqrt{1+R^2}}{p_o} \ln\left(\frac{R+\sqrt{R^2+1}}{\sqrt{\left(\frac{F}{r_o}\right)^2 + 1} - \frac{F}{r_o}} \right) \right) \tag{4.9}$$

where r is the distance along the beam axis of the transducer, F is the focal length, s_o is the nonlinear propagation parameter given by,

$$s_o = \frac{bwp_o r_o}{rc^3} \tag{4.10}$$

and R is a normalized distance from the focus given by

$$R = \frac{r-F}{r_o} \tag{4.11}$$

Furthermore, p_o is the pressure at the focus linearly extrapolated from the expected value at low amplitudes (i.e., nonlinear effects neglected), and r_o is a characteristic distance given by

$$r_o = \frac{pa^2}{l} \tag{4.12}$$

where a is the radius of the beam in the focal plane determined for a low amplitude setting. As an aside, *Bacon* [1984] never explicitly defined his beam radius, but it seems that a was intended to be the radial distance at which the pressure amplitude has decayed by e^{-1}. Taking this to be the definition and applying the field equations provided by [*Kino*, 1987], the value of a should theoretically be given by

$$a = 0.8224\, l(f/\#) \tag{4.13}$$

where $f/\#$ is the focal length of the transducer divided by its diameter, and l is the wavelength.

Before proceeding with the analysis to obtain an expression for the pressure at the focus, *Bacon* [1984] defines a transducer gain G to provide insight into some of the terms in Equation (4.9). Notice that in the absence of nonlinear effects, the ratio of the pressure at the focus (R = 0) to the pressure at center of the transducer $\left(R = -F/r_o\right)$ is given by

$$G = \sqrt{\left(\frac{F}{r_o}\right)^2 + 1} \tag{4.14}$$

which *Bacon* [1984] then defines as the gain of the transducer. More will be said about this gain parameter later in this section. Substituting Equation (4.14) into Equation (4.9) and solving for the field at the focus (R = 0) yields

$$p(F,t) = p_o \sin\left(wt - \frac{wF}{c} + \frac{s_o p}{p_o} \ln\left(G + \sqrt{G^2 - 1}\right)\right) \tag{4.15}$$

At this point in his analysis, Bacon makes a series of approximations to obtain an expression for the amplitude of the pressure wave at the focus of the transducer. Recall, that p_o in the above expressions is the pressure at the focus in the absence of nonlinear effects. Unfortunately, this quantity cannot be measured. Therefore, the first step is to determine the value of the peak pressure amplitude with nonlinear propagation included. In order to do this, *Bacon* [1984] states the following false relation without proof or any justifying argument:

$$\frac{p_m}{p_o} = \begin{cases} 1, & s \le \dfrac{p}{2} \\ \sin\left(s\,\dfrac{p_m}{p_o}\right), & \dfrac{p}{2} < s\,\dfrac{p_m}{p_o} < p \end{cases} \tag{4.16}$$

where p_m is the peak pressure amplitude at the focus and

$$s = s_o \ln\left(G + \sqrt{G^2 - 1}\right) \tag{4.17}$$

Based on Equation (4.16), Bacon defines a new propagation parameter s_m to evaluate the amount of nonlinearity in the acoustic signal given by

$$\begin{aligned} s_m = \frac{p_m}{p_o} s &= \frac{p_m}{p_o} \frac{bwp_o r_o}{rc^3} \ln\left(G + \sqrt{G^2 - 1}\right) \\ &= \frac{bwp_m F}{rc^3} \frac{\ln\left(G + \sqrt{G^2 - 1}\right)}{\frac{F}{r_o}} \\ &= \frac{bwp_m F}{rc^3} \frac{\ln\left(G + \sqrt{G^2 - 1}\right)}{\sqrt{G^2 - 1}} \end{aligned} \tag{4.18}$$

Now that the theoretical basis for *Bacon's* [1984] propagation parameter, s_m, has been summarized, the next step in evaluating its use as an indicator for nonlinearity is to discuss how it can be determined from measured pressure data at the focus of a transducer. Recall that all of Bacon's analysis neglected the off-axis field variations that would produce asymmetric distortion in the waveform. As a result, there is some uncertainty in p_m, the peak pressure amplitude at the focus. One could use the peak compressional pressure p_c, the peak rarefractional pressure p_r, or the average peak pressure $p_{avg} = (p_c + p_r)/2$, as was proposed by *Bacon* [1984]. All of these pressure values would be different. However, since the average peak pressure is not as effected by asymmetric distortion, p_{avg} should be the closest to the p_m intended by Bacon's theory.

Traditionally, the only other parameters required by Equation (4.18) that posses some uncertainty is the gain of the transducer G. In *Bacon's* [1984] derivation, G was the ratio of the amplitude at the focus to the amplitude of the intersection of the beam axis with the transducer surface. Unfortunately, this value cannot be measured explicitly. Bacon attempts to provide a method for determining G by stating that the ratio of the aperture area of the transducer to the area of the beam at the focus should be G^2. One problem is that although the aperture area of the transducer is clearly defined, the area of the beam at the focus difficult to define quantitatively. Bacon says to use the area over which the pressure amplitude of the pulse is greater than e^{-1} times the value at the focus. Unfortunately, in order to obtain an accurate value for G using this definition, the aperture area of the transducer should also be scaled to correspond to the area over which the pressure amplitude of the pulse is greater than e^{-1} times the value at the intersection point in the plane of the transducer. Bacon does not do this in his analysis. As a result, Bacon's method for determining G will always overestimate the true G value for the transducer. Also, the scaling factor for the aperture area of the transducer can only be determined analytically similar to how a was found in Equation (4.13). Therefore, there is nothing to be gained by performing the area measurement, and in our analysis, G will simply be calculated by solving for the necessary variables in (4.12), (4.13), and (4.14), yielding

$$G = \sqrt{\left(\frac{F}{p\,l\left(0.8224\cdot\left(f\ /\#\right)\right)^2}\right)^2 + 1} \tag{4.19}$$

In our work with pulsed fields, we used the wavelength corresponding to the maximum frequency of the acoustic pulse at the focus for l.

Now that we have discussed how s_m will be determined, we can evaluate its quantitative relationship to our ability to perform linear extrapolation. Since *Bacon's* [1984] theory neglects asymmetric distortion, only the error in the peak average pressure can be predicted. Substituting σ_m back into Equation (4.16) and solving for the error yields,

$$\%Error\_p_{avg} = 100\cdot\begin{cases} 0 & s_m \le p/2 \\ \dfrac{1-\sin\left(s_m\right)}{\sin\left(s_m\right)} & p/2 < s_m < p \end{cases} \tag{4.20}$$

Notice that just like in the case of Ostrovskii/Sutin's propagation parameter, s_s, there ideally exists a simple relationship between s_m and the linear extrapolation error. However, it is not clear how this relation will be effected by asymmetric distortion.

At this point, we can summarize some of the advantages and disadvantages of σ_m as an indicator of nonlinearity. First of all, s_m has traditionally been used as a nonlinear indicator. As a result, it has the advantage of being already accepted by the technical community. Secondly, s_m has the advantage of a simple relationship to the expected error in linear extrapolation at least for the case of determining the average pressure. Unfortunately, s_m also suffers from its neglecting of pulse asymmetry resulting in a difficulty in determining the proper choice for p_m, as well as some uncertainty as to how well s_m will correspond to linear extrapolation errors in p_c and p_r. Furthermore, s_m depends on the transducer gain G, a parameter that is difficult to determine quantitatively. However, both p_m and G can be defined as was done in this section if consistency is enforced in Bacon's derivations.

4.1.4 Field sigma s_z

As was mentioned in the previous section on Bacon's propagation parameter σ_m, one of the disadvantages of s_m is its dependence on the transducer gain G. In an effort to avoid this disadvantage some have proposed a modified absolute indicator of nonlinearity, denoted s_z, where the G dependence has been ignored:

$$s_z = \frac{bwp_m F}{rc^3} \tag{4.21}$$

Notice that Equation (4.21) is the same as the final formula in Equation (4.18) for s_m with the G terms removed. This oversimplification of s_m places the entire basis for s_z in a rather precarious position. It would seem highly unlikely for s_z to properly reflect the amount of nonlinear distortion in a waveform for all possible transducer gains. Furthermore, s_z lacks any sort of relationship to the expected error in linear extrapolation since it is not based on any formal derivation. However, due to its simplicity and the fact that blatant errors do not always drastically change the final result in engineering, s_z will still be considered as a possible nonlinear indicator. Of course, s_z would still have all of the disadvantages of s_m in addition to the problems introduced by the oversimplification.

4.1.5 Second harmonic ratio

For the previous proposed indices, the amount of nonlinearity in the waveform at the focus was determined based on pressure measurements made of the time domain waveform. However, nonlinear propagation also alters the frequency spectrum of the wave. As was mentioned in Chapter 3 on nonlinear absorption as well as in other references [e.g., *Hamilton and Blackstock*, 1998; *Naugolnykh and Ostrovsky*, 1998; *Pierce*, 1991], as a wave propagates in a nonlinear medium, energy will be transferred out of the fundamental frequency and into the higher harmonics. Therefore, a possible measure for the amount of nonlinearity in a wave should capture this change in the spectrum. One indicator that does this is the second harmonic ratio given by

$$H_{II} = \frac{|P(w_2)|}{|P(w_1)|} \tag{4.22}$$

where

$$P(w) = FFT\left(p(t)\right) \tag{4.23}$$

In these equations, *p(t)* is the time domain pressure waveform at the focus, w_1 is the principle frequency of the sound pulse, and w_2 corresponds to the peak in the spectrum at approximately $2w_1$ as illustrated by Figure 4.1. Notice that in these plots, w_2 is not exactly $2w_1$. This can be attributed to higher attenuation at the higher frequencies [*Pierce*, 1991].

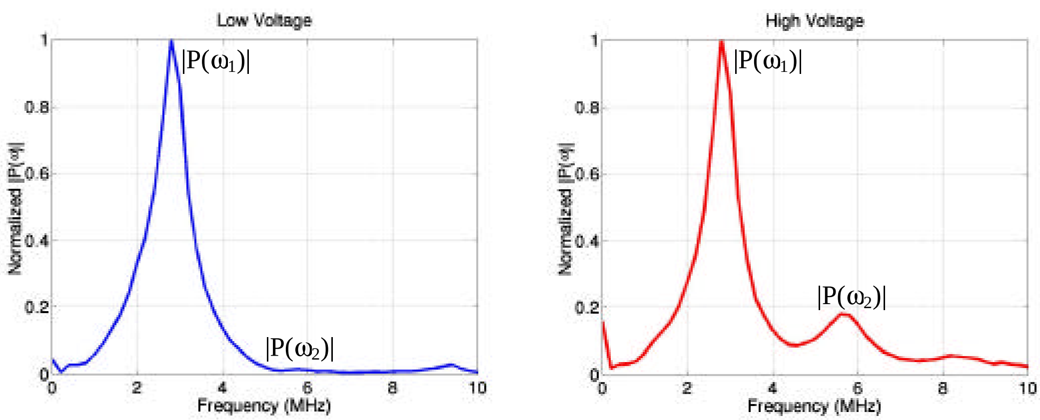

Figure 4.1: Typical frequency spectrum at the focus of a spherically focused transducer under two different drive conditions.

Due to its basis, the second harmonic ratio has many advantages. First, H_{II} has the advantage of being directly obtainable from the measured data. There is no additional dependence on estimated parameters like the transducer gain, *G*, and region boundary, R_o, nor is there any dependence on an approximate field theory as there was for Bacon's propagation parameter s_m and Ostrovskii/Sutin's propagation parameter s_s, respectively. In this sense, the second harmonic ratio is similar to the asymmetric ratio presented in Section 4.1.1. However, unlike the asymmetric ratio that could be influenced by the initial symmetry in the waveform, H_{II} should not be significantly affected by any type of biasing in the drive conditions. This is because thickness mode resonator transducers do not radiate at twice their fundamental frequency. Also, the current transducers being developed with modified frequency characteristics for harmonic imaging may be capable of receiving at the second harmonic frequency, but they are still designed not to radiate at this frequency since this would degrade the performance of harmonic imaging [*Takeuchi et al.*, 2001; *Shen and Li*, 2001].

In terms of disadvantages, there is no quantitative relationship to the expected error in linear extrapolation since the analysis is conducted in the frequency domain. It might be possible to estimate the amount of error by selecting either Ostrovskii/Sutin or Bacon's theory, determining how the harmonics would be generated for a pure sinusoidal signal, and then use these estimates to predict the error, but this entire process would be cumbersome. Furthermore, it would take away from the simplicity of H_{II} as a nonlinear indicator. Another possible disadvantage evident in Figure 4.1 is that for small drive conditions, it may be difficult to find the peak corresponding to w_2. In the worst case, the computer program would mistakenly find w_2 at the same location as w_1. However, this would only be caused by the peak at w_2 reducing to a negligible amount corresponding to insignificant nonlinear distortion. Therefore, even in this case, the second harmonic ratio calculation could be intelligently adapted to avoid numerical errors.

4.1.6 Absolute spectral index

Another possible nonlinear indicator that would reflect the generation of harmonics by nonlinear propagation is the absolute spectral index. The absolute spectral index is defined as

$$si = \frac{\int_{w_a}^{\infty} |P(w)|dw}{\int_{0}^{\infty} |P(w)|dw} \qquad (4.24)$$

where, once again, $P(w)$ is the Fourier transform of the pressure waveform at the focus. The frequency w_a is an arbitrary frequency that is selected in the hope of only including the frequencies generated by nonlinear propagation in the numerator integral of the equation. For our analysis, we considered two possible choices for w_a: $2w_1$ and $1.5w_1$, where w_1 is the location of the main frequency peak as described in Figure 4.1. Ideally, the *si* would be zero in the absence of nonlinear effects, and then grow as higher harmonics were generated by nonlinear propagation.

In terms of advantages and disadvantages, the absolute spectral index is similar to the second harmonic ratio. Like H_{II}, the *si* has the advantage of being directly measurable, once w_a has been selected, and does not depend on an approximate field theory. It also shares the disadvantage of no quantitative relationship to the expected error in linear extrapolation. However, unlike the second harmonic ratio, the absolute spectral index is not immune to biasing in the drive conditions. For example, different transducers will have different bandwidths and will be driven by pulses with varying duration. As the bandwidth and pulse duration of the generated acoustical signal varies, the frequency content above w_a will also change. The variation may be small, but it would still influence the calculated value of *si* causing the measured waveform to appear more or less nonlinear and corrupting the use of *si* as a guideline for linear extrapolation. Another disadvantage of the absolute spectral index not shared by the second harmonic ratio that was alluded to earlier is the necessity to choose an arbitrary w_a. Since w_a is arbitrary, the performance of the *si* may vary for different choices of w_a. This variability introduces some uncertainty in the evaluation of *si* as an indicator for nonlinearity because it is not possible to test every possible value of w_a.

4.2 Relative Indicators of Nonlinearity

In Section 4.1, some absolute indicators for nonlinearity were discussed. However, it may not be possible to quantify the amount of nonlinearity present in a waveform based on the current waveform alone. Therefore, in this section two indicators are proposed that also bring in information from a low voltage measurement at the focus. These indicators are the relative spectral index and the relative focal pressure. Relative indicators have the advantage that they are immune to any biasing by the initial pulse shape, but this comes at the expense of increasing the number of measurements required. In our analysis, the low voltage measurement was always taken to be the lowest voltage measurement for a particular data set.

4.2.1 Relative spectral index

The relative spectral index, as the name implies, attempts to reflect changes in the spectrum of the acoustical signal. The relative spectral index is given by

$$rsi = \frac{\int_0^{\infty} \left(\left| P_{current}(w) \right| - G_v \left| P_{low}(w) \right| \right)^2 dw}{\int_0^{\infty} \left| P_{current}(w) \right|^2 dw} \tag{4.25}$$

where G_v is an amplification factor expressing the difference in the drive conditions between the low voltage reference pressure, P_{low}, and the current focal pressure, $P_{current}$. G_v would be the same factor selected to perform linear extrapolation. Unlike the second harmonic ratio and the absolute spectral index which focus on the generation of higher harmonics, the relative spectral index has the luxury of capturing all changes in the spectrum including the generation of sub-harmonics [*Hamilton and Blackstock*, 1998] and the nonlinear absorption that was discussed in Chapter 3. Some other advantages of the *rsi* include the fact that it is directly measurable from the acoustical signals/drive voltages and there is no need to define an w_a, as was done for the case of the absolute spectral index.

Along with its theoretical advantages, the relative spectral index also has several disadvantages. First, since the calculation is done in the frequency domain, there is no quantitative relationship to the expected error in linear extrapolation. Another disadvantage of the *rsi* is that changes in the time domain may not translate to consistent changes in the frequency domain. Although the other frequency-based indicators would be susceptible to this, the *rsi* would experience the greatest sensitivity since it captures all changes in the spectrum.

4.2.2 Relative focal pressure

The last nonlinear indicator that will be evaluated in this thesis is the relative focal pressure. The goal of the relative focal pressure is to directly use the error in the linear extrapolation as an indicator for the nonlinearity. To this end, we selected the following expressions corresponding to the error in the compressional p_c, rarefractional p_r, and average p_{avg} peak pressures, respectively.

$$
\begin{aligned}
P_{ce} &= 100 \cdot \left| \frac{p_{c\,current} - G_v\, p_{c\,low}}{p_{c\,current}} \right| \\
P_{re} &= 100 \cdot \left| \frac{p_{r\,current} - G_v\, p_{r\,low}}{p_{r\,current}} \right| \\
P_{ae} &= 100 \cdot \left| \frac{p_{avg\,current} - G_v\, p_{avg\,low}}{p_{avg\,current}} \right|
\end{aligned}
\tag{4.26}
$$

In this equation, $p_{.current}$ refers to the value for the current waveform, $p_{.low}$ is the value of the reference waveform, and G_v is the voltage-based gain factor to be used in the linear extrapolation. Clearly, the relative focal pressure is based on a simple theory and is explicitly measurable. However, in order to be valid, the error in the linear extrapolation must be a monotonic function. If the extrapolation error does not always increase with increasing drive voltage, then the relative focal pressure would not always reflect the true amount of nonlinearity present in the waveform.