CHAPTER 3

THEORETICAL ANALYSIS OF NONLINEAR ABSORPTION

In the last chapter, we went through a detailed description of asymmetric pulse distortion. However, this is only one of the nonlinear mechanisms that corrupt voltage-based linear extrapolation. Nonlinear absorption is another process associated with nonlinear propagation that leads to a change in the amplitude of the propagating wave. Typically, nonlinear absorption is discussed in terms of the acoustical saturation process for plane waves [*Beyer*, 1974; *Hamilton and Blackstock*, 1998; *Pierce*, 1991]. In this chapter, we shall begin with a discussion of the saturation process and then apply the same physical principles to the fields of a focused sound source.

3.1 Acoustical Saturation for Plane Waves

The propagation of a finite amplitude plane wave, to first order, is governed by the Burgers equation [*Hamilton and Blackstock*, 1998].

$$\frac{\partial p}{\partial x} - \frac{d}{2c^3}\frac{\partial^2 p}{\partial t^2} = \frac{b}{rc^3}\frac{\partial p}{\partial t} \tag{3.1}$$

In this equation, r is the density, c is the sound speed, d is the sound diffusivity (i.e. viscosity term), and b is the traditional coefficient of nonlinearity (i.e., $1+\frac{B}{2A}$) for the fluid in which the wave propagates. Also, p is the pressure, x is the distance the wave has propagated in the medium, and t is the coordinate for the retarded time frame (i.e., $t = t - x/c$). If we assume the medium is lossless, the viscosity term drops out, and we are left with

$$\frac{\partial p}{\partial x} = \frac{b}{rc^3}\frac{\partial p}{\partial t} \tag{3.2}$$

Equation (3.2) can then be solved by the method of characteristics to yield [*McOwen*, 1996; *Hamilton and Blackstock*, 1998]

$$p(x,t) = f\left(t + \frac{bp(x,t)}{rc^3}x\right) \tag{3.3}$$

where *f(t)* is the time waveform of the acoustical signal at the source (i.e., $x = 0$). Equation (3.3) is valid until it predicts a multivalued function (i.e., a shock forms in the propagating waveform) [*McOwen*, 1996]. For the purpose of our analysis, we shall assume that the source is producing a sinusoidal waveform with an initial amplitude of p_o. Therefore, Equation (3.3) can be rewritten as

$$p(x,t) = p_0 \sin\left(wt + \frac{wbp(x,t)}{rc^3}x\right) \Rightarrow p_n(x,t) = \sin\left(wt + \frac{wbp_o}{rc^3}xp_n(x,t)\right) \tag{3.4}$$

where $p_n(x,t) = \frac{p(x,t)}{p_o}$.

Notice that in Equation (3.4) for any set distance *x*, the positive portions of the wave will occur sooner in t, and the negative portions of the wave will occur later in t. In effect, the higher-pressure values will travel faster than the lower pressure values [*Hamilton and Blackstock*, 1998]. As a result, at some propagation distance, the compressional pressure will overtake the rarefractional pressure and a shock will form [*Hamilton and Blackstock*, 1998]. The distance at which this occurs is known as the *shock formation distance* and is given by [*Hamilton and Blackstock*, 1998]

$$\bar{x} = \frac{rc^3}{wbp_o} \tag{3.5}$$

Often when studying nonlinear propagation for a plane wave, all the distances are normalized with respect to this distance [*Hamilton and Blackstock*, 1998]. Rewriting Equation (3.4) based on this normalization yields

$$p_n(x,t) = sin(wt + sp_n(x,t)) \tag{3.6}$$

where $s = \frac{x}{\bar{x}}$. The shock formation for plane wave propagation is illustrated in Figure 3.1 for successive values of s.

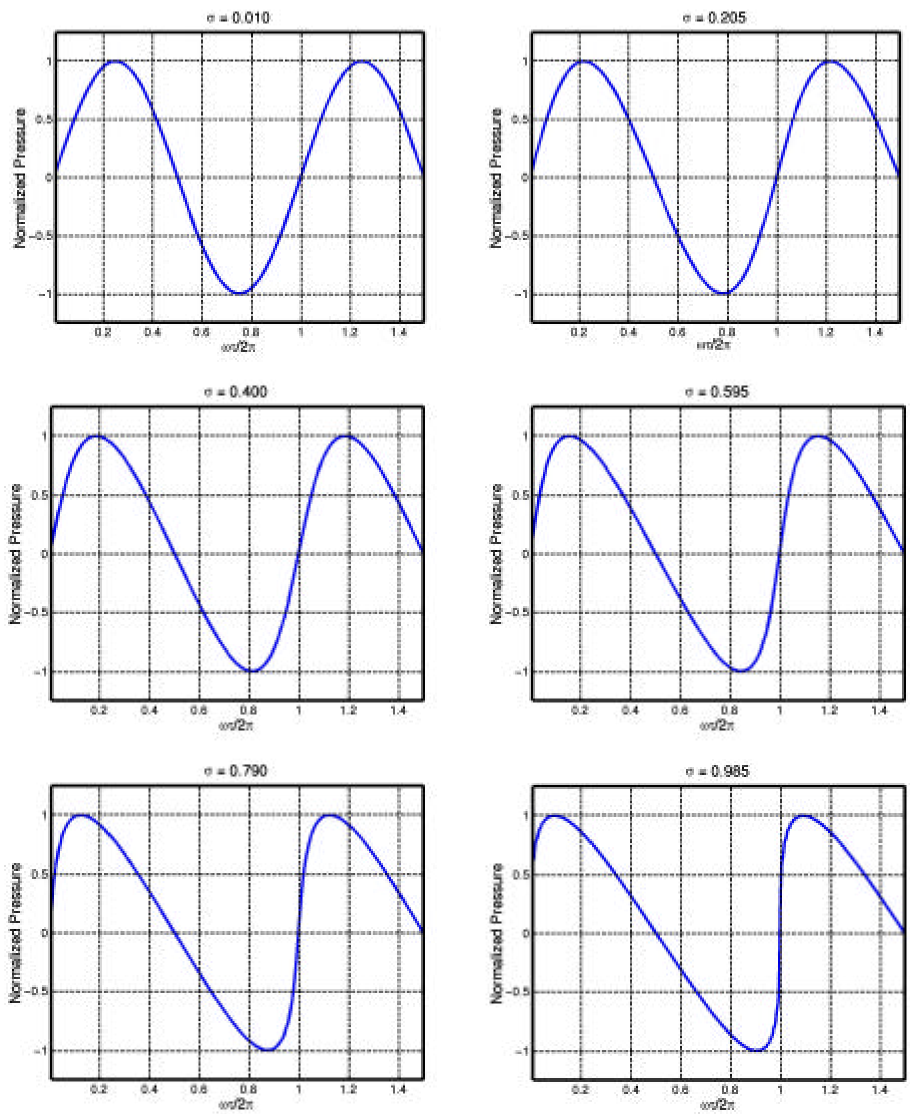

Figure 3.1: Plots illustrating shock formation for a propagating plane wave.

Notice that in these plots, the wave is almost a pure sinusoid for low s values. However, as s increases, the wave is more distorted, and by a s value of 0.985 the compressional portion of the wave has almost overtaken the rarefractional portion of the wave forming a shock. The curves shown in Figure 3.1 were generated using Fubini's solution as discussed in [*Hamilton and Blackstock,* 1998].

Once a shock is formed in the propagating wave, Equation (3.4) is no longer valid because it would predict a multivalued waveform, which is not physical. Therefore, in order to keep the solution physical, a finite amount of frequency dependent loss, known as *nonlinear absorption,* is included at the shock front [*Hamilton and Blackstock,* 1998]. As the wave propagates, energy gets transferred out of the principle frequency and into the higher harmonics. As more energy enters the harmonics, the wavefront steepens as shown in Figure 3.1. However, the frequency-dependent loss absorbs the higher frequencies at the shock so that the wave cannot fold over on itself [*Hamilton and Blackstock*, 1998]. As a result, the loss keeps the solution physical. Fortunately, there is always this type of frequency-dependent loss in real fluids.

Acoustical saturation is directly tied to the nonlinear absorption, as can be seen by restricting our attention to some point in space, x_o. The amplitude of the pressure field passing through this point will increase linearly as the source pressure is increased provided that a shock has not formed prior to x_o. If a shock has formed, then energy will have been transferred into the higher harmonics which are just absorbed by the nonlinear absorption at the shock. Increases in the source pressure contribute to increases in absorption by the shock, as well as to increases in the wave amplitude at x_o. Therefore, the pressure at x_o will be less than would be expected based on linear extrapolation. In fact if s at x_o is large enough, the pressure at x_o will be independent of the initial pressure at the source. This is known as *acoustic saturation* [*Hamilton and Blackstock*, 1998].

3.2 Nonlinear Absorption for a Focused Sound Source

Now that the basic ideas behind nonlinear absorption have been presented, we can extend these ideas to converging sound beams. The study of saturation for focused sources has been pursued by many people including [*Sempsrott,* 2000; *Duck*, 1999; *Bacon*, 1984]. However, in our analysis, we shall back away from saturation, and consider qualitatively how nonlinear absorption could be applied to some of our results from Chapter 2.

Recall that in Chapter 2 we found the following expressions for the peak compressional and rarefractional pressures at the focus of a transducer.

$$p_c = \frac{a \cdot \sin(a)}{p} \frac{2 p_o F w}{c} \frac{1}{1 - s_s}$$
$$p_r = \frac{a \cdot \sin(a)}{p} \frac{2 p_o F w}{c} \frac{1}{1 + s_s} \qquad (3.7)$$

Notice that in these expressions, as we approach shock formation (i.e., $s_s \to 1$), the compressional pressure should go to infinity, which is not a physical solution. However, in order for the peak of p_c to become sharper, energy must be transferred out of the fundamental and into higher harmonics. These higher harmonics should then be absorbed due to nonlinear absorption by the same physical arguments that govern acoustical saturation. Therefore, nonlinear absorption will serve to keep the solution physical for the focused source as well.