CHAPTER 2

THEORETICAL ANALYSIS OF ASYMMETRIC DISTORTION

One of the nonlinear effects that limit our ability to do voltage-based linear extrapolation is asymmetric distortion. Asymmetric distortion results from the combined effects of nonlinearity and diffraction. The only attempt to obtain closed form expressions for this distortion in the literature has been done by *Ostrovskii and Sutin* [1975] with *Sutin* [1978] further elaborating on the theory. Since then, several researchers have questioned the quantitative results of their work [*Lucas and Muir*, 1983; *Bacon*, 1984], but no one has proposed a more complete or alternative analysis. In this chapter, we shall work through a modified version of Ostrovskii and Sutin's method, discuss some inherent flaws in their analysis, and then discuss some insights provided by their method.

2.1 The Modified Ostrovskii/Sutin Method

The Ostrovskii/Sutin method is based on the assumption that the propagation of the acoustic wave can be decomposed into two regions [*Ostrovskii and Sutin*, 1975; *Sutin*, 1978; *Naugolnykh and Ostrovsky*, 1998]. The regions are illustrated in Figure 2.1.

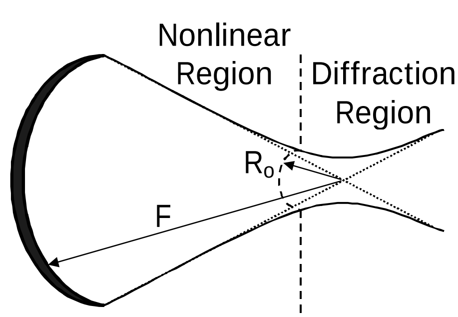

Figure 2.1: Diagram illustrating different regions of propagation used in the Ostrovskii/Sutin Method.

Away from the focus, the wave is treated as a converging nonlinear wave, and diffraction effects are ignored. Near the focus, where diffraction effects "dominate," nonlinear propagation is neglected. The boundary between the two regions of propagation is taken to be an abrupt transition occurring at some distance R_o from the focus [*Naugolnykh and Ostrovsky*, 1998]. Possible quantitative values of R_o will be discussed later in Section 2.2. The method also assumes that the medium in which the waves propagate is lossless.

In order to develop the Ostrovskii/Sutin method, let us assume that a spherically symmetric pressure wave has been generated at the surface of the circularly focused transducer which, when written in spherical coordinates, has the form

$$p(r = F, q, f, t) = p_o \sin(w\, t) \tag{2.1}$$

where F is the focal length of the lens, p_o is the initial source amplitude, and w is the frequency of the source. The spherically symmetric assumption is only a rough approximation since, as was explained in Chapter 1, the transmission coefficient of the acoustic lens would always cause the amplitude of the pressure wave to taper off as the angular distance from the axis, q, increased. However, for the purpose of this analysis, this tapering will be neglected as it was in the analysis done by Ostrovskii and Sutin who also assumed a spherically symmetric waveform at the surface of the transducer [*Ostrovskii and Sutin*, 1975; *Sutin*, 1978; *Naugolnykh and Ostrovsky*, 1998].

Neglecting diffraction effects in the first region, the pressure wave at the source would propagate according to the spherical Burger's equation given by [*Hamilton and Blackstock*, 1998]

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)p + \frac{b}{r c}\frac{\partial^2 p^2}{\partial t^2} = 0 \tag{2.2}$$

In this equation, c is the speed of sound, r is the density, and b is the traditional coefficient of nonlinearity for the medium (i.e., $1 + \frac{B}{2A}$). The viscosity term was not included in Equation (2.2) since we are assuming that the medium is lossless. Equation (2.2) when coupled with drive term given in Equation (2.1) yields a solution of the form

[*Hamilton and Blackstock*, 1998; *Sutin*, 1978; *Ostrovskii and Sutin*, 1975; *Naugolnykh and Ostrovsky*, 1998]

$$p(r,t)=\frac{p_o F}{r}\sin\left(\omega\, t+\omega\,\frac{(r-F)}{c}+\frac{p(r,t)\cdot \rho\beta\omega}{\rho c^3}\ln\left(\frac{F}{r}\right)\right) \tag{2.3}$$

provided that no shock fronts form in the converging wave. A brief analysis of shock waves is provided in Chapter 3. Equation (2.3) will be used to describe the wave propagation in the nonlinear "diffraction free" region. Under these conditions, the pressure wave at the edge of the first propagation region will satisfy Equation (2.3) with $r = R_o$. Also, the wave will remain spherically symmetric across the aperture angle α of the source because diffraction effects are ignored. The diffraction of this wave as it propagates in the diffracting "linear" region to the focus will be analyzed next.

When the wave enters the diffracting "linear" region of propagation, it must satisfy the linear wave equation. One result that can be derived from the linear wave equation is Poisson's theorem [*Pierce*, 1991]. Poisson's theorem was given in Chapter 1 and proved in Appendix A. Poisson's theorem, when applied to this problem, states that the pressure at the focus can be determined from the spherical mean of the pressure on a spherical surface enclosing the diffracting "linear" region of propagation. Expressed mathematically, this means that the pressure at the focus is given by,

$$p(0,t_o)=\left[\left(\frac{\partial}{\partial R_o}+\frac{1}{c}\frac{\partial}{\partial t}\right)R_o\cdot\bar{p}(0,R_o,t)\right]_{t=t_o-R_o/c} \tag{2.4}$$

where R_o is the radius of the spherical region, and $\bar{p}(0,R_o,t)$ is the spherical mean of the pressure over the spherical boundary given by

$$\bar{p}(0,R_o,t)=\frac{1}{4\pi\cdot R_o^{\,2}}\int_0^{2\pi}\int_{-\pi}^{\pi}p(R_o,\theta,\phi,t)\cdot R_o^2\sin(\theta)\,d\theta\, d\phi \tag{2.5}$$

However, since the wave is spherically symmetric at $r = R_o$, $\bar{p}(0,R_o,t)$ simplifies to the value of $p(R_o,t)$ given by Equation (2.3) multiplied by the ratio of the surface area of the sector covered by α to the total surface area of the sphere.

$$\bar{p}(0,R_o,t)=\frac{4\alpha\cdot\sin(\alpha)R_o^2}{4\pi\cdot R_o^{\,2}}\,p(R_o,t)=\frac{\alpha\cdot\sin(\alpha)}{\pi}\,p(R_o,t) \tag{2.6}$$

It should be mentioned that Ostrovskii and Sutin approximated the surface area of the sector covered by a to be $p\left(a R_o\right)^2$ in their work, yielding a slightly different expression for their final results [*Ostrovskii and Sutin*, 1975; *Naugolnykh and Ostrovsky*, 1998].

Now that we have derived an expression for $\bar{p}(0,R_o,t)$, we can evaluate Equation (2.4) to propagate the wave through the "diffracting" region to find the pressure waveform at the focus. In order to do this, we need to obtain expressions for the derivatives of $R_o \cdot \bar{p}(0,R_o,t)$ with respect to R_o and t. Taking the derivative with respect to R_o yields,

$$
\begin{aligned}
\frac{\partial\left(R_o\bar{p}\right)}{\partial R_o} &= \frac{a\cdot\sin\left(a\right)}{p}\,p_o F\cos\left(x\right)\left[\frac{w}{c}+\frac{bw}{rc^3}\frac{\partial}{\partial R_o}\left(pR_o\ln\left(\frac{F}{R_o}\right)\right)\right] \\
&= \frac{a\cdot\sin\left(a\right)}{p}\,p_o F\cos\left(x\right)\left[\frac{w}{c}-\frac{bw}{rc^3}\,p+\frac{bw}{rc^3}\frac{\partial\left(pR_o\right)}{\partial R_o}\ln\left(\frac{F}{R_o}\right)\right] \\
&= \frac{a\cdot\sin\left(a\right)}{p}\,p_o F\cos\left(x\right)\left[\frac{w}{c}-\frac{bw}{rc^3}\,p\right]+\frac{p_oFbw}{rc^3}\frac{\partial\left(\bar{p}R_o\right)}{\partial R_o}\ln\left(\frac{F}{R_o}\right)\cos\left(x\right) \qquad (2.7) \\
\Rightarrow \frac{\partial\left(R_o\bar{p}\right)}{\partial R_o} &= \frac{a\cdot\sin\left(a\right)}{p}\,p_o F\,\frac{\cos\left(x\right)}{1-s\cos\left(x\right)}\left[\frac{w}{c}-\overbrace{\frac{bw}{rc^3}}^{small}\,p\right] \\
&\cong \frac{a\cdot\sin\left(a\right)}{p}\,\frac{p_oFw}{c}\,\frac{\cos\left(x\right)}{1-s_s\cos\left(x\right)}
\end{aligned}
$$

where

$$s_s = \frac{p_o w F b}{rc^3}\ln\left(\frac{F}{R_o}\right) \qquad (2.8)$$

and

$$x = wt + w\,\frac{\left(R_0-F\right)}{c}+\frac{pR_0bw}{rc^3}\ln\left(\frac{F}{R_0}\right) \qquad (2.9)$$

Likewise, the derivative with respect to time can be found as,

$$
\begin{aligned}
\frac{\partial\left(R_o\bar{p}\right)}{\partial t} &= \frac{a\ \cdot \sin(a\ )}{p}\, p_o F\cos(x\ )\left[w\ \ + \frac{b w}{r c^3}\ln\!\left(\frac{F}{R_o}\right)\frac{\partial\left(pR_o\right)}{\partial t}\right] \\
&= \frac{a\ \cdot \sin(a\ )}{p}\, p_o F w\ \cos(x\ ) + \frac{p_o F b w}{r c^3}\,\frac{\partial\left(\bar{p}R_o\right)}{\partial R_o}\ln\!\left(\frac{F}{R_o}\right)\cos(x\ ) \\
&\Rightarrow \frac{1}{c}\frac{\partial\left(R_o\bar{p}\right)}{\partial t} = \frac{a\ \cdot \sin(a\ )}{p}\,\frac{p_o F w}{c}\,\frac{\cos(x\ )}{1 - s_s\cos(x\ )}
\end{aligned}
\qquad (2.10)
$$

These expressions can now be substituted into Poisson's theorem to find the pressure waveform at the focus.

$$
p\left(0, t_o\right) = \frac{a\ \cdot \sin(a\ )}{p}\,\frac{2 p_o F w}{c}\,\frac{\cos(x\ )}{1 - s_s\cos(x\ )} \qquad (2.11)
$$

Notice that Equation (2.11) varies with respect to cos*(x)*. Furthermore, since the peak compressional p_c and rarefractional p_r pressures correspond to cos*(x)* values of ±1 respectively, expressions for each of these values can be found.

$$
\begin{aligned}
p_c &= \frac{a\ \cdot \sin(a\ )}{p}\,\frac{2 p_o F w}{c}\,\frac{1}{1 - s_s} \\
p_r &= \frac{a\ \cdot \sin(a\ )}{p}\,\frac{2 p_o F w}{c}\,\frac{1}{1 + s_s}
\end{aligned}
\qquad (2.12)
$$

From these expressions, it is evident that the peak compressional pressure will be larger than the peak rarefractional pressure. Also, the peak compressional pressure should be larger than that expected by linear propagation ($s_s = 0$), and the peak rarefractional pressure should be smaller than expected. As a reminder, these expressions are only valid for s_s values less than 1 since Equation (2.3) is only valid prior to shock formation [*Hamilton and Blackstock*, 1998].

2.2 Some Inherent Problems with the Ostrovskii/Sutin Method

At this point we need to discuss some flaws in Ostrovskii/Sutin's approach to the problem. First of all, the analysis assumes that the spherical Burger's equation is valid in the diffraction free region. This would require that the wave be initially spherically symmetric across the transducer aperture, and that geometric or ray acoustics applied so that diffraction could be neglected. In order for ray acoustics to be valid, amplitude variations must be small over distances of many wavelengths [*Naugolnykh and*

Ostrovsky, 1998; *Pierce*, 1991]. However, these assumptions are mutually exclusive because if the wave is initially spherically symmetric, then there must be jump discontinuities at the boundaries of the transducer. These jump discontinuities violate the principles of ray acoustics.

Another problem with the analysis is that it assumes nonlinear effects can be neglected in the focal region. However, this is also the region where pressure amplitudes are the largest. Although diffraction effects may be more prominent near the focus, the nonlinear effects are still present. Bacon also committed on this limitation by saying that Ostrovskii/Sutin's method only produced meaningful results when the focal gain of the transducer was greater than 6π [*Bacon*, 1984]. In effect, this would set an upper limit on the size of the linear diffracting region reducing the contribution of nonlinear effects in focal region. Remember, the wave must propagate a finite difference before nonlinear effects can accumulate [*Hamilton and Blackstock*, 1998]. Bacon's limit, however, is only a partial solution. This is particularly evident when we let s_s go to 1. In this case, the compressional pressure would go to infinity regardless of the size of the focal region. This blow-up clearly does not correspond to a physical solution.

The last problem with this analysis that will be addressed in this section is that the radial boundary between the nonlinear and linear regions, R_o, is difficult to define quantitatively. This means that there is some uncertainty in the final results since σ_s has a logarithmic dependence on this value. In their original work, Ostrovskii and Sutin stated that R_o should be larger than

$$r_f = \frac{l}{1-\cos(a)} \tag{2.13}$$

They then set a limit on the gain of the transducer to insure that variations of R_o near r_f would not substantially affect the final quantitative results [*Ostrovskii and Sutin*, 1975]. As an aside, the limit on the gain they set is only stated, not proved, and it is not clear where their result comes from. From my own analysis, I would argue that their expressions are questionable at best. Based on their analysis, a good choice would be to set $R_o = r_f$.

Another possible value for R_o can be found in a later work by *Naugolnykh and Ostrovsky* [1998]. In this work, the value of R_o is selected by determining mathematically when the diffraction and nonlinear effects should contribute equally to wave propagation. These effects can be compared by performing a simple analysis on the well-known Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation given by [*Hamilton and Blackstock*, 1998]

$$\frac{c}{2}\nabla_{\perp}^{2} p = \frac{\partial^{2} p}{\partial z \partial t} - \frac{d}{2c^{3}}\frac{\partial^{3} p}{\partial t^{3}} - \frac{b}{2rc^{3}}\frac{\partial^{2} p^{2}}{\partial t^{2}} \tag{2.14}$$

where z is the along which the beam propagates, t is the slow scale transformation given by $t = t - \frac{z}{c}$, and $\nabla_{\perp}^{2}$ is the Laplacian operator applied in the *x-y* plane that governs diffraction of the beam. As an aside, the KZK expression provided by Ostrovskii and Naugolnykh in their work has a minus sign mistakenly included. However, later in their derivation this error has been removed [*Naugolnykh and Ostrovsky*, 1998].

In order to compare diffraction and nonlinearity, Equation (2.14) must be written in terms of dimensionless variables that would highlight their relative effects on the overall equation. In order to do this, define [*Naugolnykh and Ostrovsky*, 1998]

$$g = \frac{p}{p_o}, \quad \imath = wt, \quad \bar{s} = \frac{bwp_m}{rc^{3}} z = \frac{z}{L_N}, \quad h = \frac{x^{2} + y^{2}}{a^{2}} = \left(\frac{r_{\perp}}{a}\right)^{2} \tag{2.15}$$

where a is the current radius of the beam, and p_m is the current pressure amplitude. Substituting these values into Equation (2.14) yields the dimensionless form of the equation given by

$$\frac{N}{4}\nabla_{\perp}^{2} g = \frac{\partial}{\partial \imath}\left(\frac{\partial g}{\partial \bar{s}} - \frac{rwd}{2bp_o}\frac{\partial^{2} g}{\partial \imath^{2}} - g\frac{\partial g}{\partial \imath}\right) \tag{2.16}$$

In this equation, N is given by $N = \frac{L_N}{L_D}$ where L_N is the shock formation distance of a plane wave with the same amplitude propagating in the medium, and L_D is the characteristic diffraction length given by $L_D = \frac{ka^{2}}{2}$ where k is the wavenumber. Notice that if N is small, diffraction effects can be neglected [*Naugolnykh and Ostrovsky*, 1998]. In the analysis given by Ostrovskii and Naugolnykh, they also state that if N is large, nonlinear effects can be neglected for a first approximation, and then later included by adding a small perturbation at the second harmonic frequency found from further analytical analysis [*Naugolnykh and Ostrovsky*, 1998]. Ironically, there is no attempt to make such a correction in their derivation of Equation (2.12) [*Naugolnykh and Ostrovsky*, 1998].

Returning now to our initial goal of finding an appropriate value of R_o, Ostrovskii and Naugolnykh set R_o to be the value at which $N = 1$ [*Naugolnykh and Ostrovsky*, 1998]. This value can be found by considering how N will vary as the waves approach the focus. L_N is proportional to r, the distance from the origin as expressed in Equation (2.3), due to the convergence of the waves, and L_D is proportional to r^2 since the radius of the beam is proportional to r. Substituting these values into the expression for N gives

$$N(R_o) = \frac{L_N(R_o)}{L_D(R_o)} = \frac{r c^3 R_o}{b w p_o F} \cdot \frac{2F^2}{kA^2 R_o^{\,2}} = \frac{r c^3}{b w p_o} \cdot \frac{2F}{kA^2 R_o} = 1$$
$$\Rightarrow R_o = \frac{2 r c^4 F}{b w^2 p_o A^2} \qquad (2.17)$$

where A is the radius of the source. Ostrovskii and Naugolnykh give an identical expression in a slightly different form in their analysis in [*Naugolnykh and Ostrovsky*, 1998].

Before Equation (2.17) can be used to find a value for R_o, the value of the pressure at the source p_o needs to be determined. However, it normally is not practical to measure this pressure. One possibility is to determine p_o in terms of the total peak-peak pressure at the focus, $p_{p\text{-}p}$. Adding the equations in (2.12) and then solving for p_o in terms of the total peak-peak pressure yields

$$p_{p-p} = \frac{a \cdot \sin(a)}{p} \frac{4 p_o F w}{c} \frac{1}{1 - s_s^{\,2}}$$
$$\Rightarrow p_o = p_{p-p}\left(1 - s_s^{\,2}\right) \frac{c}{4Fw} \frac{p}{a \cdot \sin(a)} \qquad (2.18)$$

The value for R_o can then be expressed as

$$R_o = \frac{2 r c^4 F}{b w^2 p_o A^2} = \frac{8 r c^3 F^2}{b w A^2}\left(\frac{a \cdot \sin(a)}{p}\right)\frac{1}{p_{p-p}\left(1 - s_s^{\,2}\right)} \cong \frac{8 r c^3 F^2}{b w A^2}\left(\frac{a \cdot \sin(a)}{p}\right)\frac{1}{p_{p-p}} \qquad (2.19)$$

where $\left(1 - s_s^{\,2}\right)$ term has been removed since s_s should be appreciably smaller than 1 in order for (2.18) to be valid (i.e., ignore nonlinear effects in the focal region).

Equation (2.19) is clearly a more complicated formulation for selecting a R_o. Unfortunately, it is neither robust nor unique. For example, another valid choice for R_o could be found by choosing an N=1.5 and solving again for R_o as was done in (2.17). Therefore, the quantitative value of R_o could vary by some multiplicative constant on the order of 1, namely, the choice of $N(R_o)$, without changing the physical arguments of the derivation. Also, the formulation lacks robustness in that the total peak-peak pressure at the focus must be fairly large before R_o is small enough to approximate the size of the focal region. For example, if Equation (2.19) were to be applied to the experiment performed in the next section of the paper, the calculated value of R_o would be greater than the focal length of the transducer.

The final method to define R_o to be discussed in this paper is based on an analysis of the amplification or gain factor of a focusing acoustical source done by *Naugolnykh and Romanenko* [1959] and reviewed by *Duck* in [1999]. In their formulation, they defined a radial distance from the focus at which the amplitude of the acoustical signal found by ray acoustics is the same as the amplitude at the focus with diffraction effects included [*Duck*, 1999]. The value of this radial distance is given by [*Duck*, 1999]

$$r_f' = \frac{l}{p \sin^2(a)} \tag{2.20}$$

Although, Ostrovskii and Sutin did not consider this radius as a possible choice for R_o in their published work, this value seems to capture the desired properties of their method. Namely, this is the radius at which geometric acoustics breaks down and diffraction effects must be included for the case of linear wave propagation. Logically, the same should be true for nonlinear wave propagation.

2.3 Insights Provided by the Ostrovskii/Sutin Method

Despite the problems inherent in the Ostrovskii/Sutin method, it can still provide valuable insight into nonlinear wave propagation for a focused ultrasound transducer. First, recall that the peak compressional pressure will be larger than the peak rarefractional pressure due to the s_s in the denominator of the equations in (2.12). The s_s is placed in the denominator by taking derivatives of $R_o \cdot \overline{p}(0, R_o, t)$ (i.e., diffraction) while $\overline{p}(0, R_o, t)$ has a p dependence due to nonlinearity. Therefore, from these expressions it is clear that both diffraction and nonlinearity must be acting on a wave in order to produce asymmetric distortion.

The Ostrovskii/Sutin method also provides insight into how asymmetric distortion should develop. Recall that in the analysis, asymmetric distortion resulted from the diffraction of the nonlinear wave. Therefore, the field along the beam axis prior to the focus should have very little asymmetric distortion since diffraction effects are minimal in this region. Likewise, the asymmetric distortion should not "change" after the focal region since diffraction effects would be minimal once again. Experiments were performed in order to determine if the asymmetric distortion did vary in this manner along the beam axis.

In the experiments, a 3 MHz spherically focused transducer (Matec Instruments, Inc., Hopkinton, MA) with a diameter and focal length of 5.08 cm (*f/#* = 1) was placed opposite a PVDF membrane hydrophone (Marconi, Ltd. Essex, England) in a tank of degassed water at a water temperature of 20.1 ^{o}C. The hydrophone was then scanned in 80 μm steps along the beam axis across the focal region as described in [*Sempsrott*, 2000] for different transducer drive conditions. At each step, the pressure waveform was recorded [*Sempsrott*, 2000]. From the waveforms, the asymmetric ratio (p_c/p_r) and the total peak-peak pressure could be determined for each position along the scan. The data was then smoothed by using a sixth order polynomial to fit the data [*Sempsrott*, 2000].

Some results that nicely illustrate the development of asymmetric distortion near the focal region are shown in Figure 2.2. These results were obtained by exciting the transducer by "three cycle" voltage pulses with a center frequency of 3.09 MHz at two different amplitudes. The notation "three cycle" will be explained in Chapter 6. The lower and higher amplitude pulses are shown as solid and dashed lines, respectively. For these plots, the axial location of the maximum peak-peak pressure does not correspond to the focus because the focal length of the transducer was defined to be the location of the maximum pulse intensity integral (PII) [*Sempsrott*, 2000]. Later in the thesis, the focus will be defined as the location of the maximum peak-peak pressure.

Notice that in these experimental results, the amount of asymmetric distortion decreases steadily as we approach the source. Also, the asymmetric ratio of both the high and low drive conditions seem to be converging quickly to the same value while the total peak-peak pressure maintains roughly the same separation. This shows that asymmetry is minimal prior to the focal region.

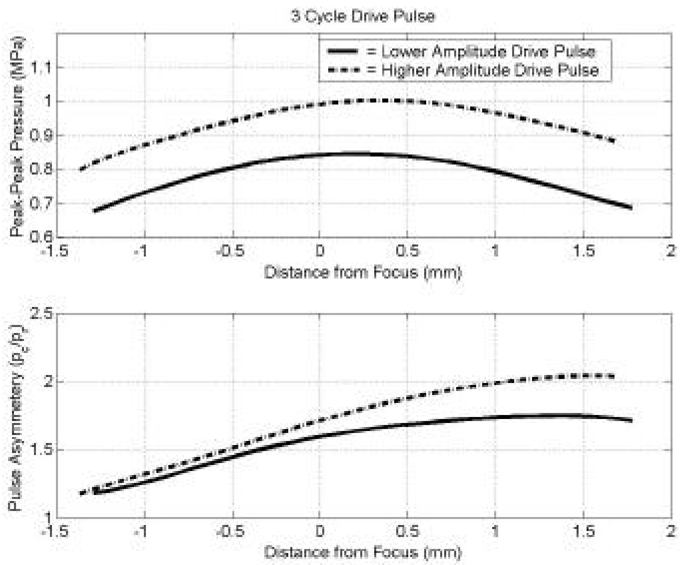

Figure 2.2: Results demonstrating pulse asymmetry near the focal region.

Likewise, the asymmetric ratio seems to saturate at some maximum value after the focus of the transducer. Once the pulse leaves the diffracting region near the focus, asymmetric distortion no longer changes the pulse shape. Therefore, there is good qualitative agreement between our experimental results and the theory predicted by the Ostrovskii/Sutin method. As a caution to those wishing to repeat these measurements, it was observed that if the scan was performed along a line slightly off from the beam axis, then the asymmetric ratio decreased after the pulse left the focal region.

The experimental results that are shown in Figure (2.2) can also provide insight into the different definitions of R_o that were provided in the previous section of this chapter. For this case, we cannot use Equation (2.19) since the resulting value for R_o would be greater than the focal length of the transducer for these power settings. Setting $R_o = r_f$ or as given by Equation (2.13) will give a boundary radius of 3.58 mm. Likewise, using $R_o = r_f'$ as given by Equation (2.20) gives a boundary radius of 0.61 mm. Based on when asymmetric distortion no longer changes the pulse shape, one would expect a R_o value on the order of 1.5 mm. Therefore, Equations (2.13) and (2.20) seem to overpredict and underpredict the value of R_o, respectively. However, a single experiment is not sufficient to completely characterize the values of the boundary radius.