APPENDIX B:
THE KLM MODEL

In this appendix, the KLM model as originally proposed by *Krimholtz et al.* [1970] is analyzed. First, an expression for the electrical input impedance of the KLM model will be derived based on the model parameters. Then, the pressure radiated by the KLM model as a function of applied voltage in the phasor domain will be determined. Finally, some related MATLAB functions are provided at the end of the appendix.

<u>Input Impedance of the KLM Model</u>

The KLM model for the piezoelectric transducer is provided in Figure B.1 [*Krimholtz et al.*, 1970].

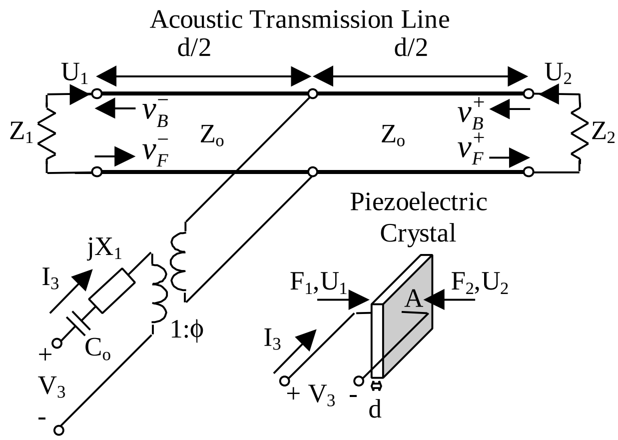

Figure B.1: The KLM model of the piezoelectric transducer.

In this model, V_3 and I_3 are the respective voltage and current applied to the piezoelectric crystal which produce the resulting acoustic forces F and particle velocities U at the respective faces of the crystal. The particle velocities inside of the crystal are denoted by $v_{F,B}^{\pm}$ where the F subscript indicates forward traveling waves propagating towards interface 2, the B subscript indicates backward-traveling waves propagating towards interface 1, and the ± denote waves in the right and left half of the crystal respectively.

The model parameters include the thickness of the crystal d, the area of the crystal A, and the characteristic impedance of the acoustic transmission line (i.e., the radiation impedance) modeling the piezoelectric crystal Z_o. The impedances Z_1 and Z_2 are the respective radiation impedances of the medium into which the crystal is radiating. A detailed discussion of acoustical impedances can be found in [*Kinsler et al.*, 2000]. In order to complete the model, it is also necessary to include a capacitor C_o, impedance jX_1, and a transformer with the ratio (1: f) that converts the electrical signal into the appropriate acoustical values. C_o results from the resonator consisting of a dielectric, the piezoelectric crystal, between two excited conducting surfaces. The values for these parameters as given by [*Krimholtz et al.*, 1970] are

$$\begin{aligned} Z_o &= rcA \\ C_o &= \frac{eA}{d} \\ X_1 &= \frac{h^2}{w^2 Z_o}\sin\left(\frac{w \cdot d}{c}\right) \\ f &= \frac{wZ_o}{2h}\operatorname{cosec}\left(\frac{w \cdot d}{2c}\right) \end{aligned} \tag{B.1}$$

where e is the permittivity of the piezoelectric under no applied voltage, h is the piezoelectric pressure constant for the crystal, r is the density, and c is the speed of longitudinal sound waves in the crystal.

Now that the model is in place, we can determine the input impedance of the piezoelectric transducer in terms of the model parameters. The impedance seen looking into port 3 is given by

$$Z_{in\_KLM} = \left(\frac{1}{jwC_o} + jX_1 + \frac{Z_a}{f^2}\right) \tag{B.2}$$

where Z_a is the impedance seen looking into the acoustic transmission line given by [*Pozar*, 1998]

$$Z_a = \frac{Z_{L1}Z_{L2}}{Z_{L1} + Z_{L2}} \tag{B.3}$$

where

$$Z_{L1,2} = Z_o \frac{Z_{1,2} + jZ_o \tan\left(\frac{w \cdot d}{2c}\right)}{Z_o + jZ_{1,2}\tan\left(\frac{w \cdot d}{2c}\right)} \tag{B.4}$$

Therefore, the total input impedance for the KLM model can be found by evaluating Equations (B.2)-(B.4).

Often a complete expression for the input impedance is not necessary, and one is only interested in obtaining an approximate expression valid near the fundamental resonant frequency for the transducer that occurs when $w = w_o = \frac{cp}{d}$. As a result, the model parameters at frequencies near w_o can be approximated as

$$\begin{aligned} X_1 &\cong \frac{h^2 p}{w^2 Z_o}\left(1 - \frac{w}{w_o}\right) \\ f &\cong \frac{w Z_o}{2h} \end{aligned} \tag{B.5}$$

Furthermore, near resonance, the impedance seen looking into the acoustical transmission line, Z_{L1} and Z_{L2}, reduce to

$$\begin{aligned} Z_{L1} &\cong \frac{Z_o^2}{Z_1} \\ Z_{L2} &\cong \frac{Z_o^2}{Z_2} \end{aligned} \tag{B.6}$$

resulting in a Z_a of

$$Z_a \cong \frac{Z_o^2}{Z_1 + Z_2} \tag{B.7}$$

Therefore the input impedance of the KLM model near resonance is given by

$$Z_{in\_KLM} \cong \left(\frac{1}{jwC_o} + \frac{jh^2 p}{w^2 Z_o}\left(1 - \frac{w}{w_o}\right) + \frac{4h^2}{w^2} \frac{1}{Z_1 + Z_2} \right) \tag{B.8}$$

Before leaving our discussion of the input impedance for the KLM model, it is important to point out that (B.8) was derived assuming that neither Z_1 nor Z_2 were zero (i.e., free space). Therefore, (B.8) cannot be applied to certain piezoelectric transducer configurations.

Radiated Pressure based on the KLM Model

In the previous section, we derived expressions for the input impedance of the piezoelectric transducer based on the KLM model. In this section, we will continue the analysis to determine the pressure radiated by the transducer when it is excited by a voltage in the phasor domain (i.e. $V_3 = |V_3(w)|e^{jq}$). The analysis will be done by summing the particle velocities at the center of the acoustical transmission line. Recall that particle velocities in the acoustical transmission line are analogous to currents in an electrical transmission line. A detailed discussion of transmission line theory is proved by *Pozar* in [1998]. Summing the velocities yields

$$\begin{aligned} v_B^+ e^{-jkd/2} - v_B^- e^{jkd/2} + \frac{I_3}{f} = 0 \\ v_F^+ e^{jkd/2} - v_F^- e^{-jkd/2} + \frac{I_3}{f} = 0 \end{aligned} \tag{B.9}$$

where k is the wave number in the crystal. However, from transmission line theory, we know that

$$\begin{aligned} v_B^+ &= G_2 v_F^+ \\ v_F^- &= G_1 v_B^- \end{aligned} \tag{B.10}$$

where G_1 and G_2 are the current transmission coefficients given by

$$\begin{aligned} G_1 &= \frac{Z_o - Z_1}{Z_o + Z_1} \\ G_2 &= \frac{Z_o - Z_2}{Z_o + Z_2} \end{aligned} \tag{B.11}$$

Substituting Equations (B.10) and (B.11) back into Equation (B.9) and solving the resulting matrix equation yields

$$\begin{aligned} v_F^+ &= \frac{I_3}{f} \frac{\left(G_1 e^{-jkd/2} - e^{jkd/2} \right)}{e^{jkd} - G_1 G_2 e^{-jkd}} \\ v_B^- &= \frac{I_3}{f} \frac{\left(e^{jkd/2} - G_2 e^{-jkd/2} \right)}{e^{jkd} - G_1 G_2 e^{-jkd}} \end{aligned} \tag{B.12}$$

Transmission line theory can then be used to solve for U_1 and U_2 yielding

$$
\begin{aligned}
U_2 &= -\frac{I_3}{f}\frac{\left(G_1 e^{-jkd/2} - e^{jkd/2}\right)}{e^{jkd} - G_1 G_2 e^{-jkd}}(1+G_2) \\
U_1 &= -\frac{I_3}{f}\frac{\left(e^{jkd/2} - G_2 e^{-jkd/2}\right)}{e^{jkd} - G_1 G_2 e^{-jkd}}(1+G_1)
\end{aligned}
\tag{B.13}
$$

Finally, replacing I_3 by V_3 and solving for the pressure wave leaving the surface of the piezoelectric crystal yields,

$$
\begin{aligned}
P_2(w) &= \frac{Z_2 V_3(w)}{f Z_{in\_KLM}}\frac{\left(G_1 e^{-jkd/2} - e^{jkd/2}\right)}{e^{jkd} - G_1 G_2 e^{-jkd}}(1+G_2) \\
P_1(w) &= \frac{Z_1 V_3(w)}{f Z_{in\_KLM}}\frac{\left(e^{jkd/2} - G_2 e^{-jkd/2}\right)}{e^{jkd} - G_1 G_2 e^{-jkd}}(1+G_1)
\end{aligned}
\tag{B.14}
$$

Equation (B.14) can also be used to determine the pressure radiated by a piezoelectric crystal excited by a transient voltage pulse by decomposing the pulse into the respective frequency components, determining the pressure radiated for each component in the Fourier domain, and then using the inverse Fourier transform to assemble the resulting pressure pulse.