APPENDIX A:
POISSON'S THEOREM

This appendix contains a brief discussion of Poisson's theorem. The theorem is stated and then proved. The statement and proof of the theorem are adapted from *Pierce* [1991]. This theorem can be easily applied to finding the pressure field at the focus of a spherically converging wave.

Statement of Theorem

Let $p(\vec{x},t)$ satisfy the wave equation for some region $\vec{x} \in \Re^3$.

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)p(\vec{x},t) = 0 \tag{A.1}$$

Also, let $\vec{x}_o$ be any point in this region. Define a sphere of radius R centered at the point $\vec{x}_o$ where R is chosen such that the medium is homogeneous inside of the sphere from some time $t_o - R/c$ to time t_o. Also, define $\bar{p}(\vec{x}_o, R, t)$ be the spherical mean of $p(\vec{x}_o + \vec{n}R, t)$ over the spherical surface given by,

$$\bar{p}(\vec{x}_o, R, t) = \frac{1}{4p \cdot R^2}\iint p(\vec{x}_o + \vec{n}R, t)dS \tag{A.2}$$

where $\vec{n}$ is the surface's outward unit normal. Then $p(\vec{x}_o, t_o)$ is given by,

$$p(\vec{x}_o, t_o) = \left[\left(\frac{\partial}{\partial R} + \frac{1}{c}\frac{\partial}{\partial t}\right)R \cdot \bar{p}(\vec{x}_o, R, t)\right]_{t=t_o - R/c} \tag{A.3}$$

Proof of Theorem:

The theorem shall be proved by operating in spherical coordinates and selecting $\vec{x}_o$ to be the origin (0,0,0). Begin by calculating the spherical mean for the full wave equation,

$$\lim_{e \to 0}\left(\frac{1}{4p \cdot R^2}\int_0^{2p}\int_e^{p-e}\left(\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)p(\vec{x},t)\right)R^2 \sin(q)dq \cdot df\right) = 0 \tag{A.4}$$

In this equation, the R^2 terms cancel. Furthermore, the ∇^2 operator in spherical coordinates is given by

$$\nabla^2 p = \frac{1}{R}\frac{\partial^2}{\partial R^2}Rp + \frac{1}{R^2\sin(q)}\frac{\partial}{\partial q}\left(\sin(q)\frac{\partial p}{\partial q}\right) + \frac{1}{R^2\sin^2(q)}\frac{\partial^2}{\partial f^2}p \qquad \text{(A.5)}$$

Substituting this expression into the above equation and simplifying where possible yields,

$$\overbrace{\frac{1}{4p}\int_0^{2p}\int_0^{p}\left[\frac{1}{R}\frac{\partial^2}{\partial R^2}Rp - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}p\right]\sin(q)\,dq\cdot df}^{I} + \overbrace{\frac{1}{4p}\int_0^{2p}\int_{q=0}^{q=p}\frac{1}{R^2}d\left[\sin(q)\frac{\partial p}{\partial q}\right]\cdot df}^{II}$$
$$+\overbrace{\lim_{e\to 0}\left[\frac{1}{4p}\int_{f=0}^{f=2p}\int_{e}^{p-e}\frac{1}{R^2\sin(q)}dq\cdot d\left[\frac{\partial p}{\partial f}\right]\right]}^{III} = 0 \qquad \text{(A.6)}$$

Now if we evaluate integrals *II* and *III* we get,

$$II = \frac{1}{4p}\int_0^{2p}\int_{q=0}^{q=p}\frac{1}{R^2}d\left(\sin(q)\frac{\partial p}{\partial q}\right)\cdot df = \frac{1}{2R^2}\left(\sin(q)\frac{\partial p}{\partial q}\right)_0^{p} = 0 \qquad \text{(A.7)}$$

because $\sin(p) = \sin(0) = 0$, and

$$\begin{aligned} III &= \lim_{e\to 0}\left(\frac{1}{4p}\int_{f=0}^{f=2p}\int_{e}^{p-e}\frac{1}{R^2\sin(q)}dq\cdot d\left(\frac{\partial p}{\partial f}\right)\right) \\ &= \frac{1}{4p\cdot R^2}\left(\frac{\partial p}{\partial f}\right)_0^{2p}\lim_{e\to 0}\Big(\left(\ln\left(\csc(q)-\cot(q)\right)\right)_e^{p-e}\Big) = 0 \end{aligned} \qquad \text{(A.8)}$$

because $\left(\frac{\partial p}{\partial f}\right)_{2p} = \left(\frac{\partial p}{\partial f}\right)_0$. This means that

$$
\begin{aligned}
I = 0 &= \frac{1}{4p}\int_0^{2p}\int_0^{p}\left(\frac{1}{R}\frac{\partial^2}{\partial R^2}Rp - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}p\right)\sin(q)dq \cdot df \\
&= \frac{1}{4p}\left(\frac{1}{R}\frac{\partial^2}{\partial R^2}R\left(\int_0^{2p}\int_0^{p} p\sin(q)dq \cdot df\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\int_0^{2p}\int_0^{p} p\sin(q)dq \cdot df\right)\right) \\
&= \frac{1}{R}\frac{\partial^2}{\partial R^2}R\left(\frac{1}{4p \cdot R^2}\int_0^{2p}\int_0^{p} pR^2\sin(q)dq \cdot df\right) \\
&\qquad - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{1}{4p \cdot R^2}\int_0^{2p}\int_0^{p} pR^2\sin(q)dq \cdot df\right) \\
&= \frac{1}{R}\frac{\partial^2}{\partial R^2}R\bar{p} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\bar{p} = 0
\end{aligned}
\tag{A.9}
$$

Now define a function *F(R,t)* by

$$
F(R,t) = \frac{\partial}{\partial R}R\bar{p} + \frac{1}{c}\frac{\partial}{\partial t}R\bar{p} \tag{A.10}
$$

and take the derivative of this function with respect to *R* and *t*:

$$
\begin{aligned}
\frac{\partial F}{\partial R} &= \frac{\partial^2}{\partial R^2}R\bar{p} + \frac{1}{c}\frac{\partial^2}{\partial R \partial t}R\bar{p} \\
\frac{\partial F}{\partial t} &= \frac{\partial^2}{\partial R \partial t}R\bar{p} + \frac{1}{c}\frac{\partial^2}{\partial t^2}R\bar{p}
\end{aligned}
\tag{A.11}
$$

Now multiply $\frac{\partial F}{\partial t}$ by $-\frac{1}{c}$ and add the result to $\frac{\partial F}{\partial R}$.

$$
\left(\frac{\partial}{\partial R} - \frac{1}{c}\frac{\partial}{\partial t}\right)F(R,t) = \frac{\partial^2}{\partial R^2}R\bar{p} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}R\bar{p} = R\left(\frac{1}{R}\frac{\partial^2}{\partial R^2}R\bar{p} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\bar{p}\right) = 0 \tag{A.12}
$$

This means that *F(R,t)* has a general solution of the form $F(R,t) = f\left(t + \frac{R}{c}\right)$. The function *f(*)* can be found in terms of $p(\vec{x},t)$ solving for *F(R=0,t)* as a function of *t*.

$$
\begin{aligned}
F(R=0,t) = f(t) &= \left(\frac{\partial}{\partial R}R\bar{p} + \frac{1}{c}\frac{\partial}{\partial t}R\bar{p}\right)_{R=0} \\
&= \bar{p} + R\left(\frac{\partial}{\partial R}\bar{p} + \frac{1}{c}\frac{\partial}{\partial t}\bar{p}\right) = \bar{p}(\vec{0},0,t) = p(\vec{0},t)
\end{aligned}
\tag{A.13}
$$

This means that

$$F(R,t) = p\left(\vec{0}, t + \frac{R}{c}\right) \Rightarrow \left(F(R,t)\right)_{t=t_o-R/c} = p\left(\vec{0}, t_o\right)$$
$$\Rightarrow p\left(\vec{0}, t_o\right) = \left(\frac{\partial}{\partial R} R\overline{p} + \frac{1}{c}\frac{\partial}{\partial t} R\overline{p}\right)_{t=t_o-R/c} \tag{A.14}$$

Since the theorem is true for $\vec{0}$, it is true for any value of $\vec{x}_o$ since a simple coordinate transformation could always be used to place any $\vec{x}_o$ at the origin.